



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>

The PUBLIC LIBRARY

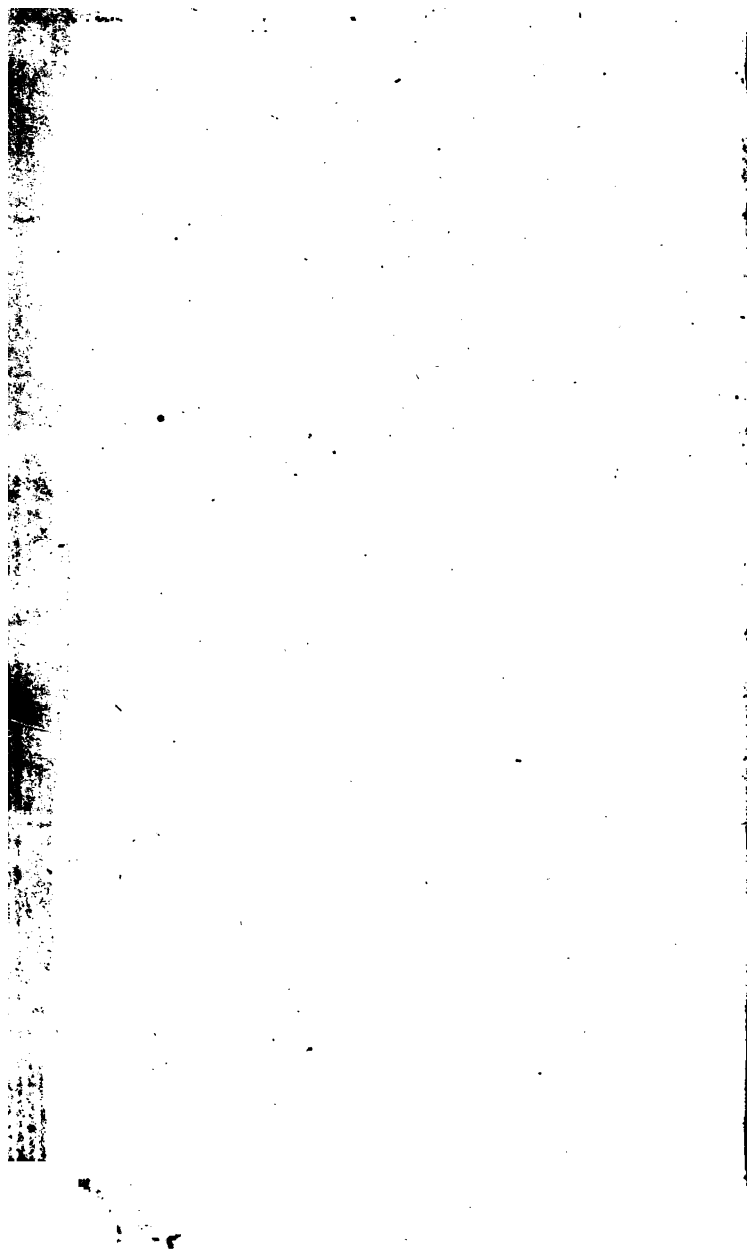
AT

Friends' School in Kendal.

Begun the 27th of the 4th mo. 1772:

N^o. 210

QA
35
.D647



THE
MATHEMATICAL
REPOSITORY.

VOL. III.

CONTAINING

Analytical Solutions

OF

A great Number of the most difficult
PROBLEMS,

RELATING TO

ANNUITIES,	INSURANCES, and
REVERSIONS,	LEASES dependent
SURVIVORSHIPS,	on Lives;

In which it has been endeavoured to exhaust the
SUBJECT.

By JAMES DODSON, F.R.S.
Accomptant and Teacher of the MATHEMATICS.

LONDON,

Printed for JOHN NOURSE, at the Lamb, opposite
Katherine Street in the Strand.

MDCCLV.

10

Hist. of science

Bowes

9-11-31 To the RIGHT HONOURABLE

24597

011-6-31MEN

GEORGE Earl of MACCLESFIELD,

P R E S I D E N T,

The COUNCIL,

And the rest of the FELLOWS of the

ROYAL SOCIETY,

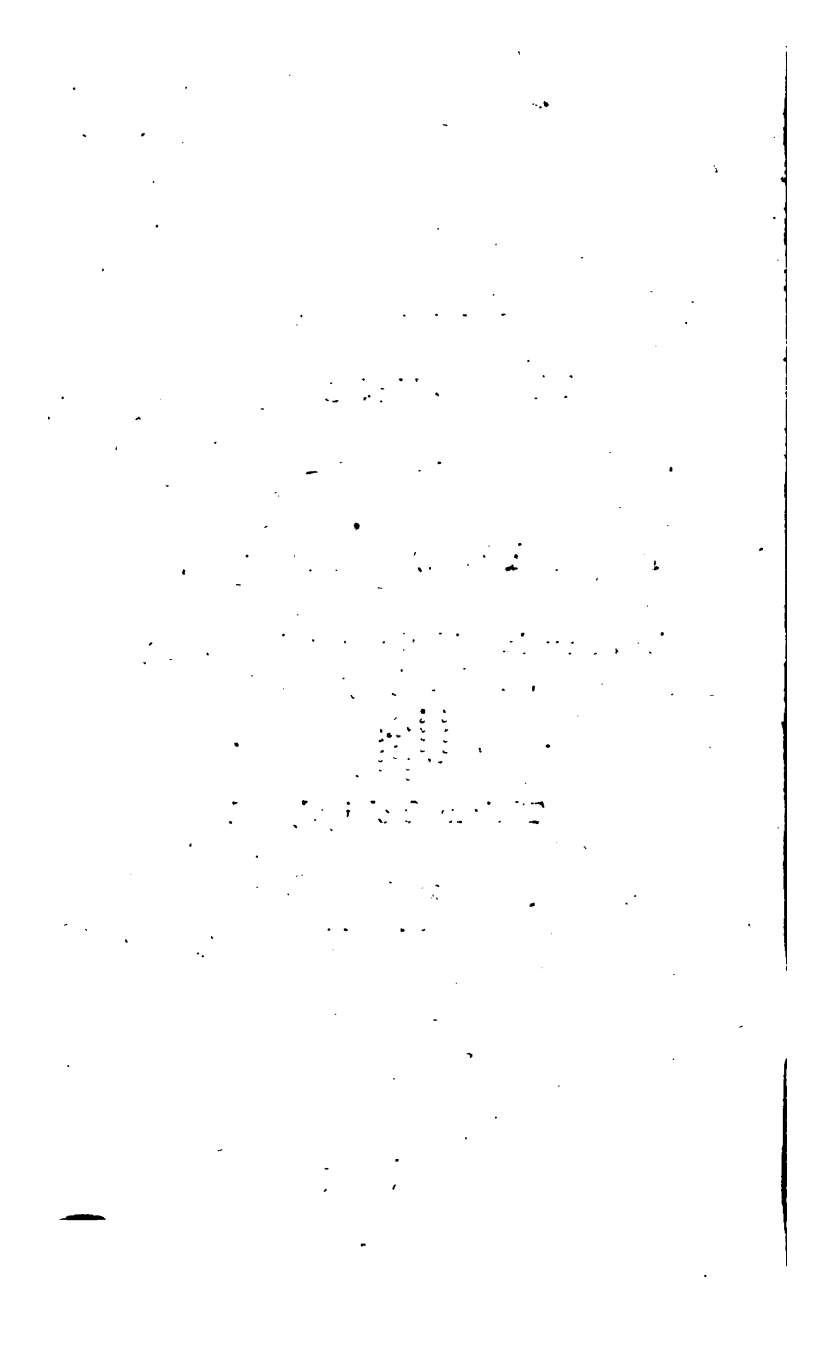
This WORK is, with the greatest Deference,
inscribed, by



Their most obliged, and

Most humble Servant,

JAMES DODSON.



THE P R E E A C E.

THE second Volume of this work being closed, as it were, with the beginning of a new subject; the reader might readily conjecture, that it was designed to prosecute it in a subsequent volume; but the author, at that time, was dubious whether it could be done, consistent with the restriction, which he had laid himself under, of introducing no process, that could not be understood by those, whose knowledge extended only to common algebra: he saw, that the doctrine of fluxions had been used, by authors of the greatest reputation, in the solution of some of the most simple of the questions, which remained to be considered; and was not sanguine enough to conceive, that he should succeed, not only in them, but also in the more complex, upon simpler principles: he saw, nevertheless, the great advantage, that would result from the solution of those questions, by a process purely arithmetical, and therefore determined to attempt it; the attempt has succeeded, even beyond his hopes, and he now submits to the approbation or censure of the public, a third volume, every question

THE PREFACE.

of which is either directly or remotely applicable to, probabilities, expectations, annuities, reversions, or insurances, dependent on lives or survivorships.

The principle, by which he has been enabled to perform this, was used in the second volume (page 341) in the investigation of the expectation of a single life; and its application, to an annuity, is equally natural; and easy; thus, conceive a person of such an age, that it cannot be reasonably expected, that he should live beyond the end of one year, suppose him possessed of an annuity of 1*l*. (or any other sum) with condition, that such part of that annual payment is to be made, to his heir, as shall be proportional to the time elapsed, between the beginning of the year and the time of his decease: then the expectation of the sum, to be so received by the heir, (being valued at the beginning of the year) will be half the annual payment, the interest of money not being here considered: the use of this principle is also extended to survivorships, with success; as will appear in the following sheets.

The solutions of the more complex questions, relating both to annuities and survivorships, would, however, have been insupportably tedious, if the method of approximation, used in the second volume, had not been again admitted; and it was so, with the greater alacrity,

THE PREFACE.

ty, because the resulting expressions are more manageable, than those in that volume.

The truly ingenious Mr. *Thomas Simpson*, in his select Exercises, pages 322 and 323; speaking of those Questions, which relate to survivorships, has the following passage. “I have, already, been very particular on these Kinds of Problems, and the more so, as there is no Method yet published (that I know off) by which They can be rightly determined. ’Tis true the Manner of proceeding by first finding the Probability of Survivorship (which Method is used in my former Work, and which a celebrated Author on this Subject has largely insisted on, in three successive Editions) may be applied to good Advantage, when the given Ages are nearly equal: But then, it is certain, that this is not a genuine Way of going to Work; and that the Conclusions hence derived are, at the best, but near Approximations. The Rate of Interest that Money bears must be compounded with the Probability of Survivorship, and the Expectation on each particular Year must be determined, in order to have a true Solution.”

The author, of this work, had been of the same opinion with Mr. *Simpson*, long before he was confirmed therein, by reading the above passage; and it is a great satisfaction to him, that he has been (almost every where) able
to

THE PREFACE.

to adhere strictly thereto, in his solutions.

The cases, relating to the order of survivorship, have been hitherto exemplified no farther, than where three lives were concerned, but are here extended to four; this, it is true, takes up a great deal of room, but then, it will be a great ease, when it shall be wanted, to find every possible case investigated; and the processes will be of service, as more examples of the manner of algebraic computation.

The subjects of copyholds, and leases of lands, for one, two, or three lives are also here discussed, in all the varieties that have occurred, either in the author's reading, practice, or imagination; and it is hoped, that some new light is thrown, on those subjects.

'Tis true, that as these do not appear till near the close of the volume, the author has been obliged (in order to prevent the swelling thereof, beyond a proper size) to abbreviate the investigations, by leaving out some intermediate steps; but as it may be reasonably supposed, that the reader will be then sufficiently master of the subject, to supply that defect, 'tis hoped, that it will be excused; especially, as the so doing will be a good test of his ability, in such kind of computation: 'tis for the same reasons, that rules, in words at length, have been omitted throughout the whole volume.

The author thinks, that little need be here
said

THE PREFACE.

said concerning insurances on lives ; as it is a subject not before handled, and will shew its own use.

There are inserted, at the end of the volume, a table of the values of annuities for single lives, secured by land ; and a table of the differences, between the values of annuities for single lives (secured by land) and the quotients arising from the divisions of the complements of those lives, by 6 times the rate of interest ; which differences occur in most of the solutions, contained in this volume.

It will appear, by the under written corrections, which the author thinks himself obliged to make, of errors committed in the second volume, that he has too much occasion to fear, that some slips have escaped him in this ; especially in those parts thereof, wherein he has ventured beyond the beaten path : he begs leave, therefore, to bespeak the candour of the judicious readers, professing himself obliged to those gentlemen that have, or shall, point out his mistakes ; and promising, publicly, to communicate and amend those, which he shall be informed of, as he now does them which are below enumerated.

In the expression given toward the bottom of page 151, Vol. II. the quantities, α and β , are severally multiplied into r ; but the multiplication was in the subsequent work of the original calculation, inadvertently, omitted :

The

THE PREFACE.

The tables of observations (pages 158 and 160) should, each of them, have been extended one line farther, in the manner following; viz. under the last printed lines of each, let lines be drawn, to denote the beginning of a new interval; then in table page 158, place 95, in the first column; 0, in the second; 0, in the third; and $+1$, in the fourth: and, in the table page 160, place 91, in the first column; 0, in the second; 0, in the third; and $+1$, in the fourth.

It happens, that these omissions affect the examples, given in pages 162 and 163, only in the second decimal place; by which means they were not discovered, before the publication of that volume; the first, was since seen, by the author, on reperforming the operation, with a gentleman who chose to study the subject: and the latter appeared, on its being discovered by another gentleman; that the values of elder lives were not given truly, by the rule: the necessary corrections follow,

In the 1st expression p. 152 for $\frac{a}{r^i}$ and $\frac{\beta}{r^{i+t}}$

read $\frac{ra}{r^i}$ and $\frac{r\beta}{r^{i+t}}$;

the second should stand, as below,

$$\frac{PPr}{a} \times \left\{ \begin{array}{l} \frac{a}{Pr} - \frac{a-b}{s} + \frac{a}{r^i} + \frac{\beta}{r^{i+t}} \\ - \frac{d}{rPr^{i+t+1}} + \frac{c-d}{vr^{i+t+1}} \end{array} \right.$$

In

THE PREFACE.

In the same page, line the 12th from the bottom, for *at the end*, read, *next following the end*.

The expression at the top of page 153 should stand thus,

$$\frac{PP_r}{a} \times \left\{ \begin{array}{l} \frac{a}{Pr} - \frac{a-b}{s} - \frac{b}{rPr^2+t \&c. + \pi} + \frac{r-b}{r^2r^2+t \&c. + \pi} \\ + \frac{a}{r^2} + \frac{\beta}{r^2+t} + \frac{\gamma}{r^2+t+v} + \frac{\delta}{r^2+t+v+w} \end{array} \right. (m-1).$$

And those, in pages 154 and 155, will be rectified, by writing $\frac{PP_r}{a}$ for $\frac{PP}{a}$; $\frac{a}{Pr}$ for $\frac{a}{P}$;

$\frac{a-b}{s}$ for $\frac{a-b}{s} \times r$; and D for Dr .

In the rule, in words at length, page 161, let the 5th and 6th paragraphs stand thus.

Multiply the number which stands in the second column of the table of observations, on a line with the given age, by the interest of one pound, and divide the product by the rate; placing the quotient under the sign +: also multiply this quotient, by the interest of 1*l*. reserving the product for future use.

Place the number which stands in the third column of the table of observations, on a line with the given age, under the sign —.

In the operation of the example, page 162, line 12, after 89, insert 95; line 15, after 79, insert 85; and line 19 after $\frac{+}{1}$, insert $\frac{+}{1}$.

That

THE PREFACE.

That part of the example contained in page 163 should stand thus

	Years	+	Years	—
Now if 524, the	2	0,9246	10	0,6756
number of persons	33	0,2741	14	0,5775
living at 10 years of	37	0,2343	19	0,4746
age, be multiplied by	44	0,1780	29	0,3207
0,04; the interest of	51	0,1353		7,0000
1/1 it produces 20,96;	60	0,0951		—
which being divided	64	0,0813		9,0484
by $(1+0,04=)$ 1,04	66	0,0751		
the rate, quotes	75	0,0528		
20,1538 which is	79	0,0451		
placed under + :	85	0,0357		
and this quotient,		20,1538		
20,1538 multiplied		—		
by 0,04, produces	sum +	22,2852		
0,80615, the divisor	sum —	9,0484		
afterwards used.		—		

Also 7, the num- 0,80615)13,2368(16,420
ber of persons dying
between the ages 10
and 11, is placed
under —.

The manner of correcting the other exam-
ple is so evident, from this, that it need not
be inserted, the true result being 17,788.

Bell Dock, Wapping,
Jan. 23d. 1755.

THE

THE MATHEMATICAL REPOSITORY.

QUESTION I.

TO find the value of an annuity, secured by a grant of lands, which is to continue during the life of a person of a given age? *

An annuity for life, secured by a grant of lands, differs from that kind of annuity for life, whose value was computed in Quest. 56. Vol. 2, in this; that here, if the annuitant dies at any time, between the stated times of the payment of the annuity, his heirs are to receive such a sum, as will be proportional to the time elapsed, between the last

* Note, all the solutions relating to annuities, reversions, survivorships, &c. in this volume, are computed upon the supposition that the decrements of life are equal; and may be adapted to any table of observations, by the application of the 104th question of the second Volume; also, in those examples where no rate of interest is mentioned, the reader is desired to compute at 4 per cent; again where the symbols t , m , n , and v , are used as the complements of life, t is always the greatest, m greater than n ; and n greater than v ; lastly A always denotes the value of that single life, whose complement is t ; and M , N , D , those of single lives whose complements are severally m , n , and v .

time of payment and the time of death; whereas, in the former case, if the annuitant dies on the day preceding the time of payment, or sooner, his heirs cannot claim any part of such payment.

SOLUTION.

If n denote the complement of life, then (by arguing as in the question above quoted) the respective probabilities of receiving the whole payment of the annuity, at the end of the first, second, third, &c. year, are $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c. besides which, in

this case, the annuitant has the expectation of receiving such part of the annuity, as shall be proportional to so much of the year, as may be elapsed before his decease, if he dies before the whole becomes due.

Now, it is an equal chance, whether the annuitant dies in the first, or second half, of any one year; and consequently, whether his heirs receive less or more than half the annual payment, growing in the year of his decease; they may therefore be esteemed, to have an expectation of half that payment, and we may (as in question 105, Vol. 2.) encrease each of the above probabilities by, $\frac{1}{2n}$, the probability of the annuitant's living half of that year in which he dies; by which means, the respective value of the first, second, third, &c. payment of the annuity will become $\left(\frac{n-1}{n} + \frac{1}{2n}, \frac{n-2}{n} + \frac{1}{2n}, \frac{n-3}{n} + \frac{1}{2n}, \&c. \text{ or } \right) \frac{2n-1}{2n}, \frac{2n-3}{2n}, \frac{2n-5}{2n}, \&c.$ the same with the expectations of life, for those years.

But the interest of money being considered, the present values of those payments will become $\left(\frac{n-1}{nr} + \frac{1}{2nr}, \frac{n-2}{nr} + \frac{1}{2nr}, \frac{n-3}{nr} + \frac{1}{2nr}, \&c. \right)$

$\frac{1}{2nr}, \frac{n-2}{nr^2} + \frac{1}{2nr^2}, \frac{n-3}{nr^3} + \frac{1}{2nr^3}$ &c. or)
 $\frac{2n-1}{2nr}, \frac{2n-3}{2nr^2}, \frac{2n-5}{2nr^3}$, &c. And consequently,
 the required value of the annuity will be represented by $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} + \frac{2n-5}{2nr^3}(n)$.

Now this differs from, $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}$
 (n) , the value of an annuity for a single life upon the
 other supposition (see quest. 56. Vol. 2.) by $\frac{1}{2nr}$
 $+ \frac{1}{2nr^2} + \frac{1}{2nr^3}(n)$; which (by quest. 15. Vol. 2.)
 is equal to $\left(\frac{1}{2n} \times \frac{1-p}{r-1} = \right) \frac{1-p}{2n} P$, where p
 denotes the present worth of one pound, due at the end
 of n years; (see table pag. 166. Vol. 2.) and P is the
 value of the perpetuity (see table pag. 77. Vol. 2.)
 $= \frac{1}{r-1}$; or the whole quantity, $\frac{1-p}{2n} P$, denotes
 the quotient arising, by dividing the present value
 of an annuity certain, for the complement of life, by
 twice that complement.

If therefore, to $P - \frac{1-p}{n} Q$ (the value of the an-
 nuity found by quest. 56, Vol. 2.) we add $\frac{1-p}{2n} P$
 (the quotient above described) the sum, $P + \frac{1-p}{2n} P -$
 $\frac{1-p}{n} Q$, will be the answer; where Q denotes
 $\left(\frac{r}{r-1} \text{ or } \right) r P P$.

If a table of the values of annuities on lives, accord-
 ing to the solution, in question 56, Vol. 2. be at hand;
 then (calling the value of a life, found by those tables,

N ; and the value required, M) then $M = N + \frac{1-p}{2n} P$; upon which principle the following tables are calculated; and may be compared with those, given in the second volume.

Scholium. The above rule (altho deduced without the assistance of Fluxions) will, upon the following trial, be found to agree (almost exactly) with one, deduced for this purpose, by Mr. *De Moivre*, from a fluxional process, in No. 473, of the philosophical transactions; which rule is preparatory to that general one, for finding the value of a life from a given table of observations, mentioned in page 18 of the preface to the second volume.

Mr. *De Moivre's* rule is expressed by $\frac{1}{r-1} - \frac{P}{an}$;

where $\frac{1}{r-1}$ signifies the same with the perpetuity (above denoted by P); a signifies the hyperbolic Logarithm of the quantity r (the amount of 1 pound and its interest for one year); and P denotes the amount of an annuity, certain, of one pound, to continue during the complement of life; which complement is there, as well as here, expressed by n .

This rule, therefore, reduced to the notation, used in this work, will become $P - \frac{1-p \times P}{an}$; for

$\left(\frac{1-p}{r-1} \text{ or } \frac{1-p}{1-p} \times P\right)$ is the value of an annuity of 1 £.

for n years certain, by quest. 15. Vol. 2.

Now, if the result above found be truly equal to that of Mr. *De Moivre*; Then,

$$P + \frac{1-p}{2n} P - \frac{1-p}{n} r PP = P - \frac{1-p}{an} P,$$

$$\text{Or } \frac{1-p}{2n} P - \frac{1-p}{n} r PP = -\frac{1-p}{an} P;$$

That is (dividing the last equation by $\frac{1-p}{n} P$)

$$\frac{1}{2} - rP = -\frac{1}{a}$$

$$\text{Or (because } P = \frac{1}{r-1}) \frac{1}{2} - \frac{r}{r-1} = -\frac{1}{a}$$

$$\text{Whence } \frac{1}{a} = \left(\frac{r}{r-1} - \frac{1}{2} \right) = \frac{r+1}{r-1 \times 2}$$

$$\text{Consequently } a = \frac{r-1 \times 2}{r+1}$$

But (by quest. 195, part 2. Vol. 1.) the hyperbolic logarithm of any number r will be represented by the series:

$$2 \times \frac{r-1}{r+1} + \frac{1}{3} \times \left[\frac{r-1}{r+1} \right]^3 \&c.$$

of which the above comparison gives the first term; which in this case is exact enough, as will appear by the following calculation.

As the interest of money has not, for many years, exceeded 5 per cent. and is continually decreasing; r will not, in any case, be expounded by a number greater than

$$(1.05 \text{ or } \frac{21}{20}); \text{ whence } r+1 = \frac{41}{20}, \&r-1$$

$= \frac{1}{20}$; Therefore $\frac{r-1}{r+1} = \frac{1}{41}$; and the above series, being expressed in numbers, will be

$$\frac{2}{41} + \frac{2}{3} \times \left[\frac{1}{41} \right]^3 + \frac{2}{5} \times \left[\frac{1}{41} \right]^5 \&c.$$

But the second term of this series, viz,

$$\left(\frac{2}{3} \times \left[\frac{1}{41} \right]^3 \right) = \frac{2}{206763} \text{ is a quantity too small}$$

to affect the calculation. When the rate of interest is less, suppose 4 per cent. the fraction whose powers constitute the series will be still smaller; for then $r =$

(1,04 or) $\frac{26}{25}$; $r + 1 = \frac{51}{25}$; and $r - 1 = \frac{1}{25}$;

whence $\frac{1}{51} = \frac{r-1}{r+1}$; and $\frac{2}{3} \times \frac{1}{51}$ will be a quantity, still smaller than the former; therefore the first term of the series will, in all cases, be sufficiently exact; and the rule above found will agree with that of Mr. De Moivre.

Notwithstanding the table, hereto annexed, of the values of annuities on lives secured by land; it is thought convenient to insert the computation of the values of those three, which have been used as examples in the questions of the second volume.

EXAMPLE I.

What is the value of an annuity (secured by land) for the life of a person, aged 43, allowing compound interest at 4 per cent.?

By example 1. quest. 56. Vol. 2. the value of an annuity for a life aged 43 years, is 12,683.

And, in this case, $P = 25$; $n = 43$; and $p = 0,1852$.

Whence $1 - p = ,8148$

Multiply by $\frac{26}{25}$

Then $(2 \times 43 = 86) 20,3700,237$: therefore $(12,683 + 0,237 =) 12,920$ will be the value required.

EXAMPLE II.

What is the value of an annuity (secured by land) for the life of a person, aged 54, at 4 per cent.?

By the 2d example to quest. 56. Vol. 2. the life aged 54 is worth 10,478. And here $P = 25$; $n = 32$; and $p = ,2851$.

Therefore $1 - p = ,7149$

Multiply by $\frac{26}{25}$

Then $(2 \times 32 = 64) 17,8725,279$; and $(10,478 + 0,279 =) 10,757$ will be the value required.

EXAM-

EXAMPLE III.

What is the value of an annuity (secured by land) for the life of a person aged 66, at 4 per cent.?

By the third example of the same question, the life of 66 is worth 7,333; and here $P = 25$; $n = 20$; and $p = .4564$

Whence $1 - p = .5436$

Multiply by $\frac{25}{25}$

Then $(2 \times 20 =) 40 \times 7,333 = 293,320$; and $(7,333 + 293,320 =) 300,653$.

Corol. 1. From Mr. De Moivre's solution of this question, may be derived a commodious method of calculating the values of these annuities, *de novo*; that is, when the values of the former kind of annuities are not given.

For it appears, that $\frac{1}{a}$ or $\frac{r+1}{r-1 \times 2}$ is a constant factor, in the negative term of the expression $(P - \frac{1-p}{a} P$ or), $P = \frac{r+1}{r-1 \times 2} \times \frac{1-p}{a} P$. If, therefore, the values of $\frac{1}{a}$ be calculated for the several rates of interest, as below, the value of the annuity will become

$$\left(P - \frac{1}{a} \times \frac{1-p}{n} P \text{ or } \right) P \times 1 - \frac{1}{a} \times \frac{1-p}{n}$$

Now, when the rate of interest is	$\left\{ \begin{array}{l} 3 \\ 3\frac{1}{2} \\ 4 \\ 4\frac{1}{2} \\ 5 \\ 6 \end{array} \right\}$	per cent.	Then	$\frac{1}{a} \left(\text{or } \frac{r+1}{r-1 \times 2} \right) =$	$\left\{ \begin{array}{l} 33,833 \\ 29,071 \\ 25,5 \\ 22,722 \\ 20,5 \\ 17,167 \end{array} \right\}$
-----------------------------------	--	-----------	------	--	--

Corol. 2. If the value of an annuity (secured by land) for a life of a given age, be required to be computed, according to a given table of observations; then find (by the rule in pag. 161. Vol. 2.) the value of the common annuity, for

that life; and (by quest. 104. Vol. 2.) find n the complement of that life; then, to the value of the annuity above found, add, $\frac{1-P}{2n} P$, the quotient above described, and the sum will be the value required.

QUESTION II.

A , whose complement is m , is possessed of an annuity for his life (secured by land) which he sells to B , for n years certain, on condition that at the expiration thereof (if A be then alive) the annuity shall return to him: the several interests of A and B in that annuity are required?

SOLUTION.

By question 57. Vol. 2, the value of an annuity (not secured by land) for n years certain, if a life whose com-

plement is m shall live so long, is $1 - \frac{1}{r^n} \times P +$

$\frac{n}{mr^n} P - 1 - \frac{1}{r^n} \times \frac{2}{m}$; To which, if $\left(1 - \frac{1}{r^n} \times \frac{P}{2m}\right)$ the sum of n terms of the series $\frac{1}{2mr} +$

$\frac{1}{2mr^2} + \frac{1}{2mr^3}$, be added, the result will (by arguing as in quest. 1.) appear to be the value of the interest of B in the annuity; viz.

$$1 - \frac{1}{r^n} \times P + 1 - \frac{1}{r^n} \times \frac{P}{2m} + \frac{n}{mr^n} P - \left(1 - \frac{1}{r^n} + \frac{2}{m},\right.$$

$$\text{Or } 1 + \frac{1}{2m} \times 1 - \frac{1}{r^n} \times P + \frac{n}{mr^n} P - \left(1 - \frac{1}{r^n} \times \frac{2}{m};\right.$$

And,

REPOSITORY.

9

And, if the above value of *B*'s interest in the annuity, be taken from $\left(P + 1 - \frac{1}{r^n} \times \frac{P}{2m} - 1 - \frac{1}{r^m} \times \frac{2}{m}\right)$ the value of the annuity for *A*'s life, the remainder will be the value of the interest of *A* in that annuity, viz.

$$\frac{1}{r^n} P + \frac{1}{r^n} - \frac{1}{r^m} \times \frac{P}{2m} - \frac{n}{mr^n} P - \left(\frac{1}{r^n} - \frac{1}{r^m} \times \frac{2}{m}\right);$$

$$\text{Or } 1 - \frac{n}{m} \times \frac{P}{r^n} - \frac{2}{m} - \frac{P}{2m} \times \frac{1}{r^n} - \frac{1}{r^m};$$

$$\text{That is } \frac{m-n}{m} \times \frac{1}{r^n} P - \frac{2}{m} - \frac{1}{2} P \times \frac{1}{r^n} - \frac{1}{r^m};$$

$$\text{Or } \frac{P}{m} \times \frac{m-n}{r^n} - r P - \frac{1}{2} \times \frac{1}{r^n} - \frac{1}{r^m}$$

$$\text{But } r P - \frac{1}{2} \left(= \frac{r}{r-1} - \frac{1}{2} = \frac{r+1}{r-1 \times 2} = \right) \frac{1}{\alpha},$$

by page 5 question 1. Therefore, if the values of $\frac{1}{\alpha}$ be taken from the table in page 7; the value of *A*'s remaining interest will become

$$\frac{P}{m} \times \frac{m-n}{r^n} - \frac{1}{\alpha} \times \frac{1}{r^n} - \frac{1}{r^m}.$$

EXAMPLE.

Suppose *A*, aged 54, would sell to *B*. his interest in such an annuity, on his own life, for 20 years certain; what ought *B* to pay for the same; and what is the value of the interest therein, which he has not disposed of?

B 5

Here

10 MATHEMATICAL

Here $P=25$; $Q=650$; $n=20$; $m=32$;

$$\frac{1}{r^n} = .4564; 1 - \frac{1}{r^n} = .5436; \text{ and } \frac{1}{r^m} = .2851.$$

Then $25 \times .5436 = 13.59$; $\frac{13.59}{2 \times 32} = .212$;
and $13.59 + .212 = 13.802$;

Now $\frac{20 \times 25 \times .4564}{32} = 7.131$; and $\frac{650 \times .5436}{32} =$

(11.042;
Therefore $(13.802 + 7.131 - 11.042 =) 9.891$ is
the value of B 's purchase.

Again, $m - n = 12$; $\frac{1}{r^n} = .2851$; and $.4564 -$
 $.2851 = .1713$;

Then $12 \times .4564 = 5.4768$; $25.5 \times .1713 = 4.3683$;

And $5.4768 - 4.3683 = 1.1085$; Therefore

$\left(\frac{1.1085 \times 25}{32} = \right) 0.866$ will be the value of what A has
not sold.

Now if 9.891, the value of B 's purchase, be added
to 0.866 the value not purchased; their sum, 10.757,
will be the value of the whole annuity.

Otherwise; if tables of the values of such annuities on
lives are at hand: then, since the numerators, $2m -$
 $2m + 1$, and $2n - 2n + 1$, of the least terms of the
two series (expressing the values of such annuities for
the lives, whose complements are m and n) are severally
unity; and since the common difference of the nume-
rators of every such series is 2; therefore the numerators
of any number of terms, whatsoever (reckoning from the
least) of both the series, which express the values of annu-
ities, for two different lives, will be severally equal:
that is, if there be two lives, whose complements are
 m , the greater, and d the lesser; then, the numerators of
the last d terms of the series, expressing the values of
the life, whose complement is m , will be $2d - 1$,
 $2d - 3$, $2d - 5$, &c. the same with those of the seri-

es, which exhibits the value of the life, whose complement is d .

For example, Let $m = 5$, and $n = 3$; then the numerators of the terms, expressing the value of the life, whose complement is $\left\{ \begin{smallmatrix} 5 \\ 3 \end{smallmatrix} \right\}$ will be $\left\{ \begin{smallmatrix} 9, 7, 5, 3, 1 \\ 5, 3, 1 \end{smallmatrix} \right\}$.

To apply this to the question before us; let us put A and B , for the values of annuities (secured by land) for two lives, whose complements are m , and $(m - n =) d$; then,

$$\text{A} = \frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} \binom{n}{1} + \frac{2d-1}{2mr^{n+1}} + \frac{2d-3}{2mr^{n+2}} \binom{d}{1} :$$

In which, $\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} \binom{n}{1}$ represents the part sold, and $\frac{2d-1}{2mr^{n+1}} + \frac{2d-3}{2mr^{n+2}} \binom{d}{1}$ the part unsold: let the value of the first be denoted by ^mA , and then that of the latter will be expressed by $\text{A} - ^m\text{A}$:

$$\text{Then } \text{A} - ^m\text{A} = \left(\frac{2d-1}{2mr^{n+1}} + \frac{2d-3}{2mr^{n+2}} \binom{d}{1} \right) = \left(\frac{1}{m \times r^n} \times \frac{2d-1}{2r} + \frac{2d-3}{2r^2} \binom{d}{1} \right) ;$$

$$\text{Therefore } m^n \times \text{A} - ^m\text{A} = \frac{2d-1}{2r} + \frac{2d-3}{2r^2} \binom{d}{1}$$

$$\text{But } \text{B} = \frac{2d-1}{2dr} + \frac{2d-3}{2dr^2} \binom{d}{1}$$

$$\text{Or } \text{B} = \frac{1}{d} \times \frac{2d-1}{2r} + \frac{2d-3}{2r^2} \binom{d}{1} ;$$

$$\text{Therefore } d \times \text{B} = \frac{2d-1}{2r} + \frac{2d-3}{2r^2} \binom{d}{1} ;$$

$$\text{Whence } m^n \times \text{A} - ^m\text{A} = d \times \text{B} ;$$

12 MATHEMATICAL

And the part unfold, viz.

$$D - {}^uM = \left(\frac{{}^uP^d}{m^n} = \right) \frac{1}{r^n} \times {}^uP \times \frac{m-n}{m};$$

And the part fold, ${}^uM = M - \frac{1}{r^n} \times {}^uP \times \frac{m-n}{m};$

EXAMPLE.

If the age be 54, and the term for which the annuity is to be sold, be 20 years.

Then M (the value of a life of 54) }
 will be 10,757 } by table
 And uP (the value of a life of (54+ }
 20=)74) = 5,057 }

Also $\frac{1}{r^n}$ (the present worth of 1 £ due at the end of 20 years) = 0,4564, by table fol. 166. Vol. 2. and

$$\frac{m-n}{m} = \left(\frac{32-20}{32} = \frac{12}{32} = \right) \frac{3}{8};$$

Therefore $(5,057 \times 0,4564 \times \frac{3}{8} =) 0,866 = D - {}^uM$, the part unfold; and $(10,757 - 0,866 =) 9,891 = {}^uM$, the part fold.

As (in the above solution) the symbol uM is put to denote the value of an annuity, certain, for n years, if a life (whose complement is m) shall continue so long; so, in the subsequent solutions, we shall use

$\left. \begin{array}{l} {}^uP \\ {}^uM \\ {}^uP \\ {}^uM \\ {}^uP \\ {}^uM \end{array} \right\}$	For the value of an annuity, for	$\left\{ \begin{array}{l} n \\ m \\ n \\ m \\ n \\ m \end{array} \right.$	Years, certain, if a life, whose complement is	$\left\{ \begin{array}{l} t \\ t \\ t \\ t \\ t \\ t \end{array} \right.$	shall live so long.
---	----------------------------------	---	--	---	---------------------

If the numeral values of the above expressions be computed at 4 per cent; and t, m, n, u , are expounded by 43, 32, 20, 10; then it will appear that,

$${}^uP = 7,228; {}^uM = 6,925;$$

$${}^uP = 10,838; {}^uM = 9,891;$$

$${}^uP = 12,528; {}^uM = 6,215.$$

Scholium

Scholium. The latter method of solution might have been applied, to quest. 57. Vol. 2. wherein is found the value of an annuity (not secured by land) for n years certain, if a life (whose complement is m) shall so long live: in which case let M and D represent the values of such annuities, for the given life, and a life n years elder than the given one; then $M - \frac{1}{r^n} \times D \times \frac{m-n}{m}$, will be the answer; differing from the above, only, in writing M and D , instead of \mathfrak{M} and \mathfrak{D} .

QUESTION III.

To find \mathfrak{P}^{ii} the value of an annuity (secured by land) to continue during the joint lives of two persons, of equal ages?

SOLUTION.

Let n be the common complement of those ages; then

$$\left(\frac{2n-1}{2n} \times \frac{2n-1}{2n} + \frac{2n-3}{2n} \times \frac{2n-3}{2n} \left(\frac{n}{2n} \right) \right. \\ \left. \text{of } \frac{n-1}{n} + \frac{1}{2n} \times \frac{n-1}{n} + \frac{1}{2n} + \right.$$

$\frac{n-2}{n} + \frac{1}{2n} \times \frac{n-2}{n} + \frac{1}{2n} \left(\frac{n}{2n} \right)$ will represent the probabilities of receiving the first, second, third, &c. payment.

$$\text{Now } \frac{n-1}{n} + \frac{1}{2n} \Big| ^2 = \frac{n-1}{n} \Big| ^2 + \frac{n-1}{n} \times \frac{1}{n} + \frac{1}{4nn}, \\ \frac{n-2}{n} + \frac{1}{2n} \Big| ^2 = \frac{n-2}{n} \Big| ^2 + \frac{n-2}{n} \times \frac{1}{n} + \frac{1}{4nn}, \\ \frac{n-3}{n} + \frac{1}{2n} \Big| ^2 = \frac{n-3}{n} \Big| ^2 + \frac{n-3}{n} \times \frac{1}{n} + \frac{1}{4nn}.$$

And these, being discounted according to the times, will give

14 MATHEMATICAL

$$\begin{aligned} \text{give } & \frac{n-1}{nr} \times \frac{n-1}{n} + \frac{n-1}{n nr} + \frac{1}{4n nr} \\ & + \frac{n-2}{nr^2} \times \frac{n-2}{n} + \frac{n-2}{n nr^2} + \frac{1}{4n nr^2} \\ & + \frac{n-3}{nr^3} \times \frac{n-3}{n} + \frac{n-3}{n nr^3} + \frac{1}{4n nr^3} \\ & \quad \&c. \quad \quad \&c. \quad \quad \&c. \end{aligned}$$

for the value of the annuity required; which, putting N for the value of the single life (as found by quest. 56. vol. 2) and Nii for the value of the equal joint lives (found by quest. 65 thereof) will become $Nii + \frac{N}{n} + \frac{1-p}{4nn}P$.

EXAMPLE.

What is the value of an annuity (secured by land) for the joint lives of two persons, each aged 48 years, allowing compound interest at 4 per cent.?

By quest. 65. vol. 2. $Nii = 8,575$ and $p = .2253$.

By table fol. 170. vol. 2. $N = 11,748$; $1-p = .7747$.

Here $n = 38$; and $P = 25$:

Then $38 \times 11,748 = 446,424$; and $4 \times 38^2 = 5776$; $19,2675(003$

Whence $(8,575 + 0,309 + 0,003) = 8,887$ is the value required.

QUESTION IV.

To approximate to the value of an annuity (secured by land) to continue, during the joint lives of two persons of given ages?

SOLUTION.

If m be the complement of the younger life, and n that of the elder; then $\frac{2n-1}{2nr} \times \frac{2m-1}{2m} +$

$\frac{1}{2n-3}$

$\frac{2n-3}{2nr^2} \times \frac{2m-3}{2m} (n)$ will represent the annuity required: and if we conceive it to consist of the products of two arithmetical progressions; viz. $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2}$ (n) whose sum is $\frac{n}{2m}$, found per table; and $\frac{2m-1}{2m} + \frac{2m-3}{2m}$

(n) whose sum is $1 - \frac{n}{2m} \times n$, by quest. 106. vol. 2.

their common differences being $\frac{1}{nr}$ and $\frac{1}{m}$; then the sum of that series will be (by quest. 21. vol. 2.)

$$\left(\frac{n \times 1 - \frac{n}{2m} \times n}{n} + \frac{n+1 \cdot 2 \cdot 3 \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{nr} \times \frac{1}{m} \right)$$

or $\frac{n}{2m} \times 1 - \frac{n}{2m} + \frac{n+1 \times n-1}{2m \times 6r}$; that is (if we

write $\frac{nn}{2m \times 6r}$ instead of $\frac{n+1 \times n-1}{2m \times 6r}$ the difference of

which expressions is, only, $\frac{1}{12mr}$) $\frac{n}{2m} \times 1 - \frac{n}{2m} +$

$$\frac{nn}{2m \times 6r}, \text{ or } \left(\frac{n}{2m} - \frac{n}{2m} + \frac{nn}{2m \times 6r} \right) = \frac{n}{2m} -$$

$\frac{n}{6r} \times \frac{n}{2m}$; which is therefore the value of the annuity required.

* In the scholium to quest. 64. vol. 2. where the method of approximating to the values of joint lives first appears, the common difference of the supposed arithmetical progression $\frac{n-1}{nr}, \frac{n-2}{nr^2}, \frac{n-3}{nr^3}$, &c. (which has the same

differences with the above) was found to be $\frac{1}{nr}$, for which reason the same common difference is here assumed.

EXAM-

16 MATHEMATICAL

EXAMPLE I.

What is the value of an annuity of 1*l.* (secured by land) for two joint lives of 43 and 54; allowing interest at 4 per cent?

Here $m = 43$; $n = 32$; $A = 10,757$ by quest. 1. and $\frac{n}{6r} = 5,128$ by tab. last. vol. 2.

From	10,757	10,757;
Take	5,128	
	<hr/>	

Rem. $5,629 \times \frac{3}{8} = 2,095,$

Remains $8,662$, the answer.

EXAMPLE II.

What is the value of an annuity of 1*l.* (secured by land) for two joint lives of 43 and 66; allowing interest at 4 per cent?

Here $m = 43$; $n = 20$; $A = 7,673$; and $\frac{n}{6r} = 3,205.$

From	7,673	7,673;
Take	3,205	
	<hr/>	

Rem. $4,468 \times \frac{20}{8} = 1,039,$

$6,634$, the answer.

EXAMPLE III.

What is the value of an annuity of 1*l.* (secured by land) for two joint lives of 54 and 66; allowing interest at 4 per cent?

Here $m = 32$; $n = 20$; $A = 7,673$; and $\frac{n}{6r} = 3,205.$

From	7,673	7,673;
Take	3,205	
	<hr/>	

Rem. $4,468 \times \frac{20}{8} = 1,039,$

$6,270$, the answer.

COROL.

COROL.

If the two lives are of equal ages, then the approximation will become $(\cancel{A} - \cancel{A} - \frac{n}{6r} \times \frac{n}{2n})$ or

$$\cancel{A} - \cancel{A} - \frac{n}{6r} \times \frac{1}{2} = \cancel{A} + \frac{n}{6r} \times \frac{1}{2}$$

Therefore the value of an annuity (secured by land) on two equal joint lives of 43, will be

$$\left(\frac{12,920 + 5,891}{2} = \right) 9,905;$$

That, of two equal joint lives of 48, will be

$$\left(\frac{12,002 + 6,089}{2} = \right) 9,046;$$

That, of two equal joint lives of 54, will be

$$\left(\frac{10,757 + 5,128}{2} = \right) 7,942;$$

And, that of two equal joint lives of 66, will be

$$\left(\frac{7,673 + 3,205}{2} = \right) 5,439.$$

SCHOLIUM.

By quest. 106. vol. 1. it appears, that the expectation of two joint lives (whose complements are m and n) may be expressed by $\frac{n}{2} - \frac{nn}{6m}$; where $\frac{n}{2}$ signifies the expectation of the single life, whose complement is n .

Now since these expressions, $\frac{n}{2}$ and $\frac{n}{2} - \frac{nn}{6m}$, are severally the sums of the two series $\frac{2n-1}{2n} + \frac{2n-3}{2n} \left(n \right)$, and $\frac{2n-1}{2n} \times \frac{2n-1}{2m} + \frac{2n-3}{2n} \times \frac{2n-3}{2m}$ (n); it follows, that the expression, $\frac{nn}{6m}$, is the sum of that series, which is composed of the differences of the terms

18 MATHEMATICAL

terms of the two former; viz. the series

$$\left(1 - \frac{2m-1}{2m} \times \frac{2n-1}{2n} + 1 - \frac{2m-3}{2m} \times \frac{2n-3}{2n} \right) (n)$$

or

$$\frac{1}{2m} \times \frac{2n-1}{2n} + \frac{3}{2m} \times \frac{2n-3}{2n} (n).$$

Again since $\frac{n}{6r}$ and $\frac{n}{6r} \times \frac{n}{2m}$ appear (by quest. 1. and this question) to be severally the sums of the two series $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} (n)$ and $\frac{2n-1}{2nr} \times \frac{2m-1}{2m} + \frac{2n-3}{2nr^2} \times \frac{2m-3}{2m} (n)$; it

follows, that, $\frac{n}{6r} \times \frac{n}{2m}$, will be the sum of that series, which is composed of the differences of the terms of those two series, viz. the series

$$\left(1 - \frac{2m-1}{2m} \times \frac{2n-1}{2nr} + 1 - \frac{2m-3}{2m} \times \frac{2n-3}{2nr^2} \right) (n)$$

or

$$\frac{1}{2m} \times \frac{2n-1}{2nr} + \frac{3}{2m} \times \frac{2n-3}{2nr^2} (n):$$

the truth of both which, the reader may render more evident, by applying the result of quest. 21. vol. 2. to the summation of them.

Now the first two series, thus compared together, viz.

$$\frac{2n-1}{2n} + \frac{2n-3}{2n} (n), \text{ and } \frac{1}{2m} \times \frac{2n-1}{2n} + \frac{3}{2m} \times \frac{2n-3}{2n} (n),$$

exhibit severally the expectations of the duration of the single life, and the differences between those and the expectations of the duration of the joint lives: and the latter pair of series, viz.

$$\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} (n), \text{ and } \frac{1}{2m} \times \frac{2n-1}{2nr} + \frac{3}{2m} \times \frac{2n-3}{2nr^2} (n)$$

exhibit the present worths of the annual payments, which are analogous to those expectations of duration: therefore whenever, in investigations of the

the expectations or probabilities of the duration of lives, we meet with either of the series, first named, or their sums $\frac{n}{2}$ and $\frac{nn}{6m}$. we may safely conclude, that the values of the annuities, corresponding to those probabilities will be expressed by the sums of the latter two series,

viz. by D and $D - \frac{n}{6r} \times \frac{n}{2m}$.

Thus, since the expectation of two joint lives is $\frac{n}{2} - \frac{nn}{6m}$; we may conclude, that an annuity (secured by land) for those joint lives will be worth $D - D - \frac{n}{6r} \times \frac{n}{2m}$; as it appeared to be, by the investigation of this question.

C O R O L.

Hence (since the expectations both of two and three joint lives, and of the longest of two or three lives, are denoted by the additions or subtractions of $\frac{n}{2}$, $\frac{nn}{6m}$,

$\frac{nnn}{12mt}$, or expressions similar thereto) it follows, that if the value of the annuity, corresponding to the expectation,

$\frac{nnn}{12mt}$ be found; then all the questions relating to annuities for two or three lives, may be answered by adding and subtracting the values of the corresponding annuities, in the same manner as the expressions are added or subtracted in the values of their expectations.

The finding the value of an annuity, which shall correspond to the expectation, $\frac{n^3}{12mt}$, requires; first, that we

discover that series of probabilities, whose sum is, $\frac{n^3}{12mt}$, the given expectation; and secondly, that the present worth of all the annual payments, expressed by those probabilities be found.

QUESTION

QUESTION V.

To find the terms of that series, of which the expression, $\frac{n^3}{12mt}$, is the sum?

SOLUTION.

By quest. 107. vol. 2. the value of the expectation of three joint lives was found to be

$\left(\frac{n}{2} - \frac{nn}{6} \times \frac{1}{m} + \frac{1}{t} + \frac{n^3}{12mt} \text{ or } \right) \frac{n}{2} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{n^3}{12mt}$; which whole expression is, therefore the sum of the series,

$$\frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} + \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t} (n).$$

Now it has been above remarked, that

$$\frac{n}{2} = \frac{2n-1}{2n} + \frac{2n-3}{2n} + \frac{2n-5}{2n} (n); \text{ and}$$

$$\frac{nn}{6m} = \frac{1}{2m} \times \frac{2n-1}{2n} + \frac{3}{2m} \times \frac{2n-3}{2n} (n);$$

And consequently, that

$$\frac{nn}{6t} = \frac{1}{2t} \times \frac{2n-1}{2n} + \frac{3}{2t} \times \frac{2n-3}{2n} (n);$$

It follows, therefore, that (if x , y , and z , represent the first, second, third, &c. term of the required series.)

$$\text{Then } \frac{2n-1}{2n} = \frac{1}{2m} \times \frac{2n-1}{2n} - \frac{1}{2t} \times \frac{2n-1}{2n} + x \\ \left(= \frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t}; \right.$$

$$\text{and } \frac{2n-3}{2n} = \frac{3}{2m} \times \frac{2n-3}{2n} - \frac{3}{2t} \times \frac{2n-3}{2n} + y \\ \left(= \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t}; \right.$$

Also

$$\text{Also } \frac{2n-5}{2n} - \frac{5}{2m} \times \frac{2n-5}{2n} - \frac{5}{2t} \times \frac{2n-5}{2n} + x$$

$$\left(= \frac{2n-5}{2n} \times \frac{2m-5}{2m} \times \frac{2t-5}{2t} \right);$$

&c. &c.

Which equations being transposed, and severally divided by $\frac{2n-1}{2n}, \frac{2n-3}{2n}, \frac{2n-5}{2n}$ &c. will become,

$$\frac{2n}{2n-1} x = \left(\frac{2m-1}{2m} \times \frac{2t-1}{2t} - 1 + \frac{1}{2m} + \frac{1}{2t} \right)$$

$$\left(\frac{1}{4mt}; \right.$$

$$\frac{2n}{2n-3} y = \left(\frac{2m-3}{2m} \times \frac{2t-3}{2t} - 1 + \frac{3}{2m} + \frac{3}{2t} \right)$$

$$\left(\frac{9}{4mt}; \right.$$

$$\frac{2n}{2n-5} z = \left(\frac{2m-5}{2m} \times \frac{2t-5}{2t} - 1 + \frac{5}{2m} + \frac{5}{2t} \right)$$

$$\left(\frac{25}{4mt}; \right.$$

$$\text{For } \left\{ \begin{array}{l} \frac{2m-1}{2m} \times \frac{2t-1}{2t} = \frac{4mt-2m-2t+1}{4mt}; \\ \frac{2m-3}{2m} \times \frac{2t-3}{2t} = \frac{4mt-6m-6t+9}{4mt}; \\ \frac{2m-5}{2m} \times \frac{2t-5}{2t} = \frac{4mt-10m-10t+25}{4mt}. \end{array} \right.$$

$$\text{And } \left\{ \begin{array}{l} 1 - \frac{1}{2m} - \frac{1}{2t} = \frac{4mt-2m-2t}{4mt}; \\ 1 - \frac{3}{2m} - \frac{3}{2t} = \frac{4mt-6m-6t}{4mt}; \\ 1 - \frac{5}{2m} - \frac{5}{2t} = \frac{4mt-10m-10t}{4mt}. \end{array} \right.$$

Then

22 MATHEMATICAL

Then, if from the three former fractions, these three latter be taken; the remainders will be $\frac{1}{4mt}$, $\frac{9}{4mt}$, and $\frac{25}{4mt}$, as was above asserted.

$$\text{Whence } x = \left(\frac{2n-1}{2n} \times \frac{1}{4mt} \right) \frac{2n-1}{2n} \times \frac{1}{2m} \times \frac{1}{2t};$$

$$y = \left(\frac{2n-3}{2n} \times \frac{9}{4mt} \right) \frac{2n-3}{2n} \times \frac{3}{2m} \times \frac{3}{2t};$$

$$z = \left(\frac{2n-5}{2n} \times \frac{25}{4mt} \right) \frac{2n-5}{2n} \times \frac{5}{2m} \times \frac{5}{2t};$$

Therefore the required series will be

$$\frac{2n-1}{2n} \times \frac{1}{2m} \times \frac{1}{2t} + \frac{2n-3}{2n} \times \frac{3}{2m} \times \frac{3}{2t} + \left(\frac{2n-5}{2n} \times \frac{5}{2m} \times \frac{5}{2t} \right) (-n).$$

If the reader investigates the sum of this series, by quest. 22. vol. 1. (and in the result, writes m for $n+1 \times n-1$), the sum thereof will appear to be $\frac{n^3}{12mt}$, which operation will be a farther confirmation of the truth of this process.

QUESTION VI.

To find the present value of the annual payments, expressed by the above series; or, in other words, to find the sum of the series $\frac{2n-1}{2nr} \times \frac{1}{2m} \times \frac{1}{2t} +$

$$\frac{2n-3}{2nr^2} \times \frac{3}{2m} \times \frac{3}{2t} \left(n \right)?$$

SOLU.

SOLUTION.

Here the sums of the three separate series, whose products compose the above, are $\frac{nn}{2m}$, $\frac{nn}{2t}$; their common differences, $\frac{1}{nr}$, $-\frac{1}{m}$,* and $-\frac{1}{t}$; and their first terms, $\frac{2n-1}{2nr}$, $\frac{1}{2m}$, and $\frac{1}{2t}$: therefore (by quest.

$$22. \text{ vol. 2.}) \text{ the sum will be } \left(\frac{1}{nn} \times \frac{nn}{2m} \times \frac{nn}{2t} \right. \\ \left. + \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{2n-1}{2nmtr} - \frac{1}{2mnt} - \frac{1}{2ntmr} \right. \\ \left. - \frac{n+1 \cdot n \cdot n-1^2}{2 \cdot 2 \cdot 2} \times \frac{1}{nmtr} \right); \\ \text{or } \left(\frac{nn}{4mt} + \frac{n+1 \cdot n-1 \cdot 2n-3}{4 \cdot 6 \cdot mtr} - \frac{n+1 \cdot n-1^2}{4 \times 2mtr} \right);$$

In which (writing nn for $n+1 \times n-1$) we shall have

$$\frac{nn}{4mt} + \frac{nn}{4mt} \times \frac{2n-3}{6r} - \frac{nn}{4mt} \times \frac{n-1}{2r};$$

$$\text{But } \frac{2n-3}{6r} - \frac{n-1}{2r} = \left(\frac{2n-3-3n+3}{6r} \right) = -\frac{n}{6r};$$

Therefore $\left(\frac{nn}{4mt} - \frac{nn}{4mt} \times \frac{n}{6r} \right) = \frac{nn}{4mt} \times \frac{5r-n}{6r}$ will be the value required.

* In the solutions of questions 21 and 22. vol. 2. the arithmetical progressions were assumed to be decreasing; and their common differences were, notwithstanding, deemed affirmative; now, in this operation, there are two increasing progressions, and their common differences are considered as negative, because otherwise, the result will be the same, as would have arose from decreasing progressions.

QUESTION VII.

A and *B*, (whose complements of life are *t* and *m*) have an annuity (secured by land) for their joint lives; which they will sell to *C*, for *n*, a number of years, less than either of those complements; on condition, that (if they both survive that period) the annuity shall return to them again; the value of *C*'s interest in that annuity is required?

SOLUTION.

If *t* be greater than *m*; then (by question 4,) $\frac{2m-1}{2mr} \times \frac{2t-1}{2t} + \frac{2m-3}{2mr^2} \times \frac{2t-3}{2t} \left(\frac{m}{2t} \right)$ will represent the whole annuity; therefore $\frac{2m-1}{2mr} \times \frac{2t-1}{2t} + \frac{2m-3}{2mr^2} \times \frac{2t-3}{2t} \left(\frac{n}{2t} \right)$ will represent that part thereof proposed to be sold to *C*.

Now, if this be conceived to consist of the products of the terms of two arithmetical progressions, viz.

$\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} \left(\frac{n}{2t} \right)$ whose sum is

$\left(\frac{1}{m} \times \frac{m-n}{r^n} - \frac{1}{\infty} \times \frac{1}{r^n} - \frac{1}{r^m} \right)$ or $\frac{1}{r^n}$, by

quest. 2. and $\frac{2t-1}{2t} + \frac{2t-3}{2t} \left(\frac{n}{2t} \right)$ whose sum is

$1 - \frac{n}{2t} \times n$, by quest. 106. vol. 2. their common

differences being $\frac{1}{mr}$ and $\frac{1}{t}$; then the sum of that series will (by question 21. vol. 2.) be

$$\left(\frac{1}{r^n} \times 1 - \frac{n}{2t} \times n \right) \frac{1}{2 \cdot 2 \cdot 3} \times \frac{1}{mr^n};$$

Or)

Or) $n \cdot m \times 1 - \frac{n}{2t} + \frac{n + 1 \cdot n \cdot n - 1}{2t \cdot 6rm}$; that is
 (writing nn for $n + 1 \cdot n - 1$)

$$\left(n \cdot m - \frac{n}{2t} \cdot n \cdot m + \frac{n}{2t} \times \frac{nn}{6rm} \text{ or } \right)$$

$$n \cdot m - n \cdot m - \frac{nn}{6rm} \times \frac{n}{2t}.$$

EXAMPLE.

What is the value of an annuity (secured by land) for 20 years certain, if two persons of the ages 43 and 54, shall both live so long; allowing compound interest at 4 per cent?

Here $n = 20$; $m = 32$; $t = 43$; $\frac{n}{6r} = 3,205$
 and $n \cdot m$ (by exa. 1. quest. 2.) = 9,891.

Whence $\frac{nn}{6rm} = \frac{3,205 \times 20}{32} = 2,003$; and $9,891$
 $- 2,003 = 7,888$:

Also $\frac{7,888 \times 20}{2 \times 43} = 1,835$; Th. $(9,891 - 1,835 =)$
 8,056 will be the value required.

COROL. I.

If the two persons, on whose joint lives the annuity is held, are of equal ages: then $t = m$; and the value re-

quired will become $n \cdot m - n \cdot m - \frac{nn}{6rm} \times \frac{n}{2m}$.

COROL. II.

If the complement of one of the two persons, on whose joint lives the annuity is held, be equal to the number of years for which the annuity is proposed to be sold; then

$n \cdot m = m$; and $m = n$; whence the value required

will become $\left(\frac{n}{6r} \times \frac{n}{2t} \text{ or } \frac{n}{6r} \times \frac{n}{2t} \right)$ the same with the value of an annuity, on the two joint lives, whose complements are n and t .

QUESTION VIII.

To find the value of an annuity (secured on land) to continue during the joint lives of three persons of equal ages?

SOLUTION.

If n be the common complement of those lives; then

$$\left[\frac{n-1}{n} + \frac{1}{2n} \right]^3, \left[\frac{n-2}{n} + \frac{1}{2n} \right]^3, \left[\frac{n-3}{n} + \frac{1}{2n} \right]^3,$$

&c. will be the probabilities of receiving the first, second, third, &c. payment.

$$\text{Now } \left[\frac{n-1}{n} + \frac{1}{2n} \right]^3 = \left[\frac{n-1}{n} \right]^3 + 3 \times \left[\frac{n-1}{n} \right]^2 \times \frac{1}{2n} \\ \left(+ 3 \times \frac{n-1}{n} \times \frac{1}{4n} + \frac{1}{8n^3} \right);$$

$$\left[\frac{n-2}{n} + \frac{1}{2n} \right]^3 = \left[\frac{n-2}{n} \right]^3 + 3 \times \left[\frac{n-2}{n} \right]^2 \times \frac{1}{2n} \\ \left(+ 3 \times \frac{n-2}{n} \times \frac{1}{4n} + \frac{1}{8n^3} \right);$$

$$\left[\frac{n-3}{n} + \frac{1}{2n} \right]^3 = \left[\frac{n-3}{n} \right]^3 + 3 \times \left[\frac{n-3}{n} \right]^2 \times \frac{1}{2n} \\ \left(+ 3 \times \frac{n-3}{n} \times \frac{1}{4n} + \frac{1}{8n^3} \right);$$

&c.

&c.

There-

Therefore $\left(\frac{n-1}{n}\right)^3 \times \frac{1}{r} + \left(\frac{n-1}{n}\right)^2 \times \frac{3}{2nr} +$
 $\left(\frac{n-1}{n}\right) \times \frac{3}{4nr} + \frac{1}{8nr^2}$
 $+ \left(\frac{n-2}{n}\right)^3 \times \frac{1}{r^2} + \left(\frac{n-2}{n}\right)^2 \times \frac{3}{2nr^2} +$
 $\left(\frac{n-2}{n}\right) \times \frac{3}{4nr^2} + \frac{1}{8nr^3}$
 $+ \left(\frac{n-3}{n}\right)^3 \times \frac{1}{r^3} + \left(\frac{n-3}{n}\right)^2 \times \frac{3}{2nr^3} +$
 $\left(\frac{n-3}{n}\right) \times \frac{3}{4nr^3} + \frac{1}{8nr^4}$
 &c. &c. &c.

Will represent the value of the annuity required;

Or (putting N , N^{ii} , and N^{iii} for the values of 1, 2, and 3 equal joint lives, as found in questions, 56, 65, and 72, vol. 2.)

$$P^{iii} = N^{iii} + \frac{2N^{ii}}{2n} + \frac{3N}{4n} + \frac{1-p}{8n^3} P$$

COROLLARY.

Since $P = N + \frac{1-p}{2n} P$

$$P^{ii} = N^{ii} + \frac{2}{2n} N + \frac{1-p}{4n^2} P$$

$$P^{iii} = N^{iii} + \frac{3}{2n} N^{ii} + \frac{3}{4n^2} N + \frac{1-p}{8n^3} P$$

Therefore $P^{iv} = N^{iv} + \frac{4}{2n} N^{iii} + \frac{6}{4n^2} N^{ii} +$
 $\left(\frac{4}{8n^3} N + \frac{1-p}{16n^4} P\right)$

And the law of continuation is manifest.

QUESTION IX.

To approximate to the value of an annuity (secured by land) for the joint lives, whose complements are n , m and t ; n being the least, and t the greatest?

SOLUTION.

By quest. 107. vol. 2. the expectation of the three joint lives is $\frac{n}{2} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{n^3}{12mt}$; which expression consists of four terms, which are the several sums of the four series, $\frac{2n-1}{2n} + \frac{2n-3}{2n} \left(\frac{1}{2m} \right)$;

$$\frac{2n-1}{2n} \times \frac{1}{2m} + \frac{2n-3}{2n} \times \frac{1}{2m} \left(\frac{1}{2m} \right);$$

$$\frac{2n-1}{2n} \times \frac{1}{2t} + \frac{2n-3}{2n} \times \frac{1}{2t} \left(\frac{1}{2t} \right);$$

$$\text{And } \frac{2n-1}{2n} \times \frac{1}{2m} \times \frac{1}{2t} + \frac{2n-3}{2n} \times \frac{1}{2m} \times \frac{1}{2t} \left(\frac{1}{2t} \right).$$

And, if the terms of each of those series be severally multiplied by $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$, &c. then the sums of the resulting series will (by questions 1, 4, and 6.) be $\frac{1}{r}$,

$$\frac{1}{r} - \frac{n}{6r} \times \frac{n}{2m}, \frac{1}{r} - \frac{n}{6r} \times \frac{n}{2t}, \text{ and}$$

$$\frac{1}{r} - \frac{n}{6r} \times \frac{nn}{4mt}; \text{ which (being combined by the same}$$

$$\text{signs as } \frac{n}{2} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{n^3}{12mt} \Big) \text{ will give}$$

$$\frac{1}{r} - \frac{1}{r} - \frac{n}{6r} \times \frac{n}{2m} - \frac{1}{r} - \frac{n}{6r} \times \frac{n}{2t}$$

$$+ \frac{1}{r} - \frac{n}{6r} \times \frac{nn}{4mt} \text{ for the value required; which}$$

may

may be reduced to

$$(R - R - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{4mt} \text{ or})$$

$$R - R - \frac{n}{6r} \times \frac{m+t \times 2 - n \times n}{4mt}.$$

But, in order (by another proof) to render the truth of this method of proceeding, quite evident; let us investigate (by question 22. vol. 2.) the sum of the series

$$\frac{2n-1}{2nr} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} + \frac{2n-3}{2nr^2} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t}$$

(n), which exhibits the value of the joint lives required.

Here the sums of the three constituting series are, R ,

$$1 - \frac{n}{2m} \times n; \text{ and } 1 - \frac{n}{2t} \times n; \text{ their common differ-}$$

$$\text{ences, } \frac{1}{nr}, \frac{1}{m}, \text{ and } \frac{1}{t}; \text{ and first terms, } \frac{2n-1}{2nr},$$

$$\frac{2m-1}{2m}, \text{ and } \frac{2t-1}{2t}; \text{ whence their sum will be}$$

$$\left(\frac{1}{nr} \times R \times 1 - \frac{n}{2m} \times n \times 1 - \frac{n}{2t} \times n \right. \\ \left. + \frac{n+1}{2} \times \frac{n-1}{2} \times \frac{n-1}{2} \times \frac{1}{2npr} + \frac{2m-1}{2mnr} + \frac{2t-1}{2tmr} \right. \\ \left. - \frac{n+1}{2} \times \frac{n-1}{2} \times \frac{n-1}{2} \times \frac{1}{nr} \times \frac{1}{m} \times \frac{1}{t} \right)$$

$$\text{Or) } R \times 1 - \frac{n}{2m} \times 1 - \frac{n}{2t} + \\ \frac{n+1}{2} \times \frac{n-1}{2} \times \frac{n-1}{2} \times \frac{1}{2npr} - \frac{n+1}{2} \times \frac{n-1}{2} \times \frac{1}{2r} \times \frac{1}{4mt};$$

which, if we write nn for $n+1 \cdot n-1$, and expand

$$\left(1 - \frac{n}{2m} \times 1 - \frac{n}{2t} \text{ or } \frac{2m-n}{2m} \times \frac{2t-n}{2t} \text{ by multi-} \right. \\ \left. C 3 \text{ plication,} \right)$$

plication, will become $(P \times \frac{4mt - 2mn - 2tn + nn}{4mt} - \frac{nn \times n - I}{6r \cdot 4mt})$

$$+ \frac{nn \times 2n + 2mn + 2tn - 3}{6r \cdot 4mt} - \frac{nn \times n - I}{6r \cdot 4mt}$$

$$\text{Or) } P \times \frac{4mt - 2mn - 2tn + nn}{4mt} + \frac{n}{6r} \times \frac{2mn + 2tn - nn}{4mt}$$

$$\left(- \frac{n}{6r} \times \frac{3nn - 3n}{4mt} \right)$$

$$\text{That is } (P \times \frac{4mt - 2mn - 2tn + nn}{4mt} + \frac{n}{6r} \times \frac{2mn + 2tn - nn}{4mt})$$

$$\text{Or) } P \times \frac{4mt}{4mt} - \frac{2mn + 2tn - nn}{4mt} P$$

$$\left(+ \frac{2mn + 2tn - nn}{4mt} \times \frac{n}{6r} \right)$$

Which may be reduced to

$$(P - P - \frac{n}{6r} \times \frac{2mn + 2tn - nn}{4mt})$$

$$\text{Or) } P - P - \frac{n}{6r} \times \frac{m + t \times 2 - n \times n}{4mt}$$

the same result as was obtained by the other process.

Having thus established the principle mentioned, in schol. quest. 4. viz. that whenever the expectations, or the sums of the probabilities of the receiving any annuity, are denoted by $\frac{n}{2}$, $\frac{nn}{6m}$, $\frac{n^3}{12mt}$, or such like expressions, any how compounded; that, then the value of the annuity itself may be denoted by P , $P - \frac{n}{6r} \times \frac{n}{2m}$.

$P - \frac{n}{6r} \times \frac{nn}{4mt}$ or such like expressions, compounded as the former; we shall in all future solutions take the latter, for the values of such annuities, whose expectations, or the sums of whose probabilities, are expressed by the

the former; without repeating either the Argument whereon the same is founded, or the proof thereof; because thereby we shall greatly shorten our Operations, as sufficiently appears by the two processes, used in the solution of this question.

EXAMPLE.

What is the value of an annuity (secured by land) for the joint lives of three persons, aged 43, 54, and 66, allowing compound interest at 4 per cent?

Here $R = 7,674$; $n = 20$; $m = 32$; $t = 43$ and

$$\frac{n}{6r} = 3,205;$$

$m = 32$	98	$R =$	$7,674$
$t = 43$	43	$-\frac{n}{6r} =$	$3,205$
75	98	Rm^t	$4,469$
2	128		
150	1376		

$-n = 20$	4	multiply by	2600
130	$5504 = 4mt$		2681400
20			8938
2000			$7,674 = R$
			$559411619,400(2,111$
			$5,563 \text{ the ans.}$

COROL. I.

If the two youngest lives are equal, then $t = m$, and the value of the annuity will become

$$R - R - \frac{n}{6r} \times \frac{4m - n \times n}{4mn}.$$

Hence the value of an annuity (secured by land) on the joint lives of three persons, two of which are aged 54, and the other 66, allowing interest at 4 per cent. will be 5,318.

COROL. II.

If the two elder lives are equal, then $m = n$; and the value will be $\mathfrak{P} - \mathfrak{P} - \frac{n}{6r} \times \frac{2t + n}{4t}$:

And the value of an annuity (secured by land) on the joint lives of three persons, two of which are aged 66, and the third 43, allowing interest at 4 per cent, will be 4.920.

COROL. III.

If the three lives are of equal ages, then $t = m = n$; and the joint lives will become $\mathfrak{P} - \mathfrak{P} - \frac{n}{6r} \times \frac{3}{4}$:

And the value of an annuity (secured by land) for the joint lives of three persons, each aged 66, will be 4.321.

QUESTION X.

A, *B*, and *C*, whose Complements of life are t , m , and n (n being the least) have an annuity (secured by land) for their joint lives; which they would sell to *D*, for, v , a number of years less than either of their complements; on condition, that (if they all survive that period) the annuity shall return to them again: what is *D*'s purchase worth?

SOLUTION.

By arguing as before, the proposed annuity will be represented by the series $\frac{2n-1}{2nr} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t}$
 $+ \frac{2n-3}{2nr^2} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t} (v)$; which may be
 con-

conceived as constituted of the products of the terms of 3 arithmetical progressions: viz.

$\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} (v)$	whole first term is	$\frac{2n-1}{2nr}$	sum	$1 - \frac{v}{2m} \times v$	and common difference	$\frac{1}{nr}$
$\frac{2m-1}{2m} + \frac{2m-3}{2m} (v)$		$\frac{2m-1}{2m}$		$1 - \frac{v}{2m} \times v$		$\frac{1}{m}$
$\frac{2t-1}{2t} + \frac{2t-3}{2t} (v)$		$\frac{2t-1}{2t}$		$1 - \frac{v}{2t} \times v$		$\frac{1}{t}$

Therefore the sum thereof will (by quest. 22. vol.

2.) be $\left(\frac{v}{2m} \times 1 - \frac{v}{2m} \times v \times 1 - \frac{v}{2t} \times v \right)$

$$+ \frac{v+1}{2} \times \frac{v \cdot v-1}{2 \cdot 2 \cdot 3} \times \frac{2n-1}{2nrmt} + \frac{2m-1}{2mrnt} + \frac{2t-1}{2trnt}$$

$$- \frac{v+1 \cdot v \cdot v-1}{2 \cdot 2 \cdot 2 \cdot nrmt} \text{ or } \frac{v}{2m} \times 1 - \frac{v}{2m} \times 1 - \frac{v}{2t}$$

$$+ \frac{v+1 \cdot v \cdot v-1}{2 \cdot 2 \cdot 3} \times \frac{2t+2m+2n-3}{2nrnt}$$

$$- \frac{v+1 \cdot v \cdot v-1}{2 \cdot 2 \cdot 2 \cdot nrnt} ; \text{ which, by writing } vv \text{ for}$$

$$v+1 \times v-1, \text{ and expanding } \left(1 - \frac{v}{2m} \times 1 - \frac{v}{2t} \right) \text{ or}$$

$$\frac{2m-v}{2m} \times \frac{2t-v}{2t} \text{ by multiplication, will become}$$

$$\left(\frac{v}{2m} \times \frac{4mt-2mv-2tv+vv}{4mt} + \frac{v^3 \times 2t+2m+2n-3}{6r \times 4nmt} \right)$$

$$- \frac{v^3 \times v-1}{2r \times 4nmt} \text{ or } \frac{v}{2m} \times \frac{4mt-2mv-2tv+vv}{4mt}$$

$$+ \frac{vv}{6nr} \times \frac{2tv \times 2mv + 2nv - 3v - 3v \times v-1}{4mt};$$

34 MATHEMATICAL

That is $\frac{v}{6rn} \times \frac{4mt}{4mt} - \frac{v}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt}$
 $+ \frac{vv}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt} + \frac{vv}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt}$
 or) $\frac{v}{6rn} - \frac{v}{6rn} = \frac{vv}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt}$
 $+ \frac{vv}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt}$; which may be reduced to $\frac{v}{6rn}$
 $- \frac{v}{6rn} \times \frac{vv}{6rn} \times \frac{2mt + 2mt + 2mt}{4mt} \times \frac{v}{4mt}$

EXAMPLE.

What is the value of an annuity, for 10 years certain, if three joint lives, whose complements are 43, 32, and 20, should live 10 long?

Here (by quest. 2.) $v = 6,215$; $v = 10$; $\frac{v}{6rn}$
 $= 1,603$; $n = 20$; $m = 32$; $t = 43$; $\frac{vv}{6rn} = \frac{1,603 \times 10}{20}$
 $= 801.5$; $32 + 43 \times 2 + 10 = 140$; $20 - 10 \times 2 = 20$;
 Now $6,215 - 801.5 = 5,413.5$; $5,413.5 \times 140 = 757,890$;
 $801.5 \times 20 = 16,030$; and $757,890 - 16,030 = 741,860$;
 Then $\frac{741,860 \times 10}{4 \cdot 32 \cdot 43} = 1,348$; and $6,215 - 1,348$
 $= 4,870$ will be the value required.

COROL. I.

If the two youngest lives are equal, then $t = m$ and the value required will be

$$\frac{v}{6rn} - \frac{v}{6rn} = \frac{vv}{6rn} \times \frac{4m - v}{4mn} - \frac{vv}{6rn} \times \frac{2m - v}{4mn} \times \frac{v}{4mn}$$

COROL.

COROL. II.

If the two elder lives be equal, then $m=n$; but no other alteration will happen in the first given expression.

COROL. III.

If the three lives are of equal ages, then n may be wrote for m , in the expression given in *Corol. I*; but no other alteration will happen therein.

QUESTION XI.

To find $L. \mathcal{P}^{ii}$ the value of an annuity (secured by land) to continue during the longest of two equal lives?

SOLUTION.

By arguing as in quest. 76. vol. 2. if, from the sum of the values of the single lives, we take the value of the joint lives; the remainder will be the answer.

$$\text{Here } 2\mathcal{P} = 2N + \frac{1-p}{2n} 2P;$$

$$\text{And } \mathcal{P}^{ii} = N^{ii} + \frac{2}{2n} N + \frac{1-p}{4n^2} P;$$

$$\text{Therefore } L. \mathcal{P}^{ii} = 2N - N^{ii} - \frac{2}{2n} N + \frac{1-p}{2n} 2P$$

$$\left(- \frac{1-p}{4n^2} P \right)$$

$$\text{Now } 2N - N^{ii} = L. N^{ii};$$

$$\text{And let } \frac{1-p}{2n} P = A; \text{ and } \frac{1-p}{4n^2} P = \left(\frac{A}{2n} \right) = B;$$

$$\text{Then } L. \mathcal{P}^{ii} = L. N^{ii} - \frac{2}{2n} N + 2A - B.$$

36 MATHEMATICAL

The solution of this question, and those of questions the 3d and 8th, are sufficient to shew, that the exact values of annuities (secured by land) for combined lives, cannot be obtained without a calculation, much more tedious and difficult, than that of the values of such annuities, not so secured: and since the approximations to the values of the former are rather easier, than those to the latter, and equally near the truth, it hath been thought expedient, to give only the approximations to the solutions of the following questions.

QUESTION XII.

To approximate to the value of an annuity (secured by land) for the longest of two lives, whose complements are x , the lesser, and m the greater?

SOLUTION.

By quest. 108. vol. 2. the expectation of the longest of those lives is $\frac{x}{2} + \frac{xm}{6m}$; and the sum of the values of the annuities, corresponding to those expectations, is $\frac{x}{6r} + \frac{m}{6r} - \frac{x}{6r} \times \frac{x}{2m}$; the value of the annuity required.

EXAMPLE.

What is the value of an annuity (secured by land) for the longest of two lives, of the ages 54 and 66, allowing compound interest at 4 per cent?

By exam. 2. quest. 1. $\frac{x}{6r} = 10,757;$

By exam. 3. quest. 4. $\frac{x}{6r} - \frac{x}{6r} \times \frac{x}{2m} = \frac{1,397}{12,154};$

Therefore, will be the value required.

Q2

Or (according to the rule in vol. 2. page 234.)

To the value of the life of 54, viz. $\text{£} = 10,757;$

Add the value of the life of 66, viz. $\text{£} = 7,674;$

And, from the sum $18,431,$

Take the value of the joint lives (ex. 3. quest. 4.) $6,277;$

Remains the value required $12,154.$

COROL.

When the two lives are equal, then $\text{£} = \text{£}$, and $m=n$; whence the value of the annuity required will be-

come $(\text{£} + \text{£} - \frac{n}{6r} \times \frac{1}{2} \text{ or }) 3 \text{ £} - \frac{n}{6r} \times \frac{1}{2}.$

QUESTION XIII.

A and *B* who are possessed of an annuity (secured by land) for the longest of their two lives, propose to sell the same to *C*, for a number of years, less than the complement of either of their lives; on condition, that if both, or either of them, survive that period, the annuity shall revert to them again; required the value of *C*'s purchase?

SOLUTION.

Let the complements of the lives of the two annuitants be represented by t and m ; and the time for which the annuity is sold by n ; then

From the sum of the values of annuities for n years certain, on the single lives, viz.

$$n \text{ £} + n \text{ £}$$

Take the value of an annuity, for n years certain, on their joint lives (quest. 7.)

$$n \text{ £} - n \text{ £} - \frac{nm}{6rm} \times \frac{n}{2t}$$

The remainder, viz.

$$n \text{ £} + n \text{ £} - \frac{nm}{6rm} \times \frac{n}{2t}$$

will be the value of *C*'s interest in the annuity.

E X.

EXAMPLE.

What is the value of an annuity (secured by land) for twenty years certain, if either of two persons, of the ages 43 and 54, shall live so long?

Here $\frac{1}{r}$ is by question 2 10,838;

And $\frac{1}{r} - \frac{1}{6r} \times \frac{1}{2t}$ (is by quest. 7-) 1,835;

Therefore, 12,673;
will be the value required. "

QUESTION XIV.

To approximate to the value of an annuity (secured by land) for the longest of three lives, whose complements are n , m , and t .

SOLUTION.

By quest. 109. vol. 2. the expectation of those lives is $\frac{t}{2} + \frac{mm}{6t} + \frac{n^3}{12mt}$; and consequently the annuity re-

quired will be worth $(\frac{1}{r} + \frac{1}{r} - \frac{m}{6r} \times \frac{m}{2t} +$
 $(\frac{1}{r} - \frac{n}{6r} \times \frac{tn}{4ms})$

or) $\frac{1}{r} + \frac{1}{r} - \frac{m}{6r} \times m + \frac{1}{r} - \frac{n}{6r} \times \frac{nn}{2m} \times \frac{1}{2t}$.

EXAMPLE.

What is the value of an annuity (secured by land) for the longest of three lives, whose ages are 43, 54, and 66, allowing compound interest at 4 per cent?

Here

Here (by quest. 1.) $\{ \begin{array}{l} \text{A} = 10,757; \text{ and } \text{B} = 7,673; \\ \text{P} = 12,920; \end{array} \}$

$\{ \begin{array}{l} z = 43; m = 32; \\ n = 20; \end{array} \} \frac{m}{6r} = 5,128; \text{ and } \frac{n}{6r} = 3,205;$

Remr. $\frac{5,629}{4,468}$

Then $5,629 \times 32 = 180,128$; and $4,468 \times \frac{400}{64}$

$(= 27,925;$

Ans $\frac{180,128 + 27,925}{86} = \left(\frac{208,053}{86} = \right) 2,419:$

Lastly $(12,920 + 2,419 =) 15,339$ is the value required.

Or (by proceeding in the manner of the rule in quest. 80. vol. 2. page 244.)

To the value of the life of 43 }
Add that of 54 } quest. 1. } 12,920.
And that of 66 } } 10,757.
Also the value of the three joint lives (quest. 9.) 5,562.

And from their sum, 26,912;

Take the sum of 43 and 54 }
the values of the 43 and 66 } quest. 1. } 8,662
joint lives of 54 and 66 } 4. } 6,634 } 21,572;
Remains the answer 15,340

COROL. I.

If the two younger lives are equal, then $\text{P} = \text{A}$, and $z = m$; whence the above value will become

$$(\text{A} + \text{A} - \frac{m}{6r} \times \frac{m}{2m} + \text{B} - \frac{n}{6r} \times \frac{nn}{4mm} \text{ or})$$

$$\text{A} + \text{A} - \frac{m}{6r} \times \frac{1}{2} + \text{B} - \frac{n}{6r} \times \frac{nn}{4mm}$$

And the value of an annuity (secured by land) to continue as long as either of three persons (two of whom are aged 54, and the other 66) shall be alive, allowing compound interest at 4 per cent. will be 14,007. C O.

40 MATHEMATICAL

COROL. II.

If the two elder lives are equal, then $\overline{m} = \overline{n}$, and $m = n$; whence the value will be

$$(\overline{r} + \overline{n} - \frac{n}{6r} \times \frac{n}{2t} + \overline{n} - \frac{n}{6r} \times \frac{nn}{4nt} \text{ or})$$

$$\overline{r} + \overline{n} - \frac{n}{6r} \times \frac{n}{2t} + \frac{n}{4t}; \text{ that is } \overline{r} + \overline{n} - \frac{n}{6r} \times \frac{3n}{4t};$$

whence the value of an annuity (secured by land) to continue as long as either of three persons (two of whom are aged 66 and the other 43) shall be alive, allowing compound interest at 4 per cent. will be 14.478.

COROL. III.

If the three lives are of equal ages, then $\overline{r} = \overline{m} = \overline{n}$, and $r = m = n$; whence the value of an annuity on the longest of them will be

$$\overline{r} + \overline{r} - \frac{r}{6r} \times \frac{n}{2n} + \overline{r} - \frac{r}{6r} \times \frac{nn}{4nn}, \text{ or}$$

$$(\overline{r} + \overline{r} - \frac{n}{6r} \times \frac{1}{2} + \frac{1}{4} =) \overline{r} + \overline{r} - \frac{n}{6r} \times \frac{3}{4}:$$

Therefore the value of an annuity (secured by land) for the longest of three lives, each aged 66, allowing compound interest at 4 per cent. will be 11.024.

SCHOLIUM.

The value of an annuity (secured by land) to continue as long as any two, of three persons of given ages, are alive, may (by schol. 2. quest. 85; vol. 2.) be found as follows.

Let n, m, t , represent the complements, and $\overline{r}, \overline{m}, \overline{t}$, the values of the lives of the eldest, second, and youngest person; $\overline{r, m}, \overline{m, t}, \overline{r, t}$, the values of their joint lives, taken two and two; and $\overline{r, m, t}$ the value of their three joint lives. Then

$$\text{Then } \frac{1}{6r} = \frac{1}{6r} - \frac{1}{6r} - \frac{m}{6r} \times \frac{n}{2t}$$

$$\frac{1}{6r} = \frac{1}{6r} - \frac{1}{6r} - \frac{n}{6r} \times \frac{n}{2t}$$

$$\frac{1}{6r} = \frac{1}{6r} - \frac{1}{6r} - \frac{n}{6r} \times \frac{n}{2m}$$

$$-2 \times \frac{1}{6r} = -2 \times \frac{1}{6r} - \frac{n}{6r} \times \frac{n}{m} + \frac{n}{t} - \frac{nn}{2mt};$$

And the result will be

$$\left(\frac{1}{6r} - \frac{1}{6r} - \frac{m}{6r} \times \frac{n}{2t} + N - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{2mt} \right),$$

$$\text{or) } \frac{1}{6r} - \frac{1}{6r} - \frac{m}{6r} \times \frac{n}{2t} - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{2mt};$$

will be the value required.

EXAMPLE III.

What is the value of an annuity (secured by land) to continue as long as any two, of three persons, of the ages 66, 54, and 43, shall be alive?

Here $n = 20$; $m = 32$; $t = 43$; $\frac{1}{6r} = 10.757$;

$$\frac{1}{6r} - \frac{m}{6r} = 5.629; 5.629 \times 32 = 180.128; \frac{1}{6r} - \frac{n}{6r}$$

$$= 4.468; \frac{t + m - n \times n}{m} = \left(\frac{55.20}{32} \right) \frac{275}{8};$$

$$4.468 \times \frac{275}{8} = 153.875; 180.128 - 153.875 = 26.541$$

And $\frac{26.541}{2 \cdot 43} = 0.309$; therefore $(10.757 - 0.309 =)$ 10.448, will be the value required.

COROL. I.

If the two younger persons are of equal ages; then $t = m$; and,

$\frac{1}{6r} -$

$$100 - \frac{1}{2m} \times 100 - \frac{m}{6r} \times m - 100 - \frac{n}{6r} \times \frac{2m - n \times n}{m},$$

will be the value required.

COROL. II.

If the two elder persons are of equal ages; then $100 = 100$, and $m = n$; whence,

$$100 + \frac{1}{2t} \times 100 - \frac{n}{6r} \times t - n, \text{ will be the value required.}$$

COROL. III.

If the three persons are of equal ages; then the value required will be 100 .

QUESTION XV.

A, B, and C, whose complements of life are n , m , and t ; (whereof n is the least and t the greatest) being possessed of an annuity, for the longest of their three lives, agree with D, to sell him the same for v , a number of years less than either of their complements; upon condition, that if all, or any of them, be alive at the expiration of that time, the annuity shall revert to them; what is the value of D's purchase?

SOLUTION.

To $v100 + v100 + v100$, the values of annuities for v years certain, on the three single lives (by quest. 2.) add

$$v100 - v100 - \frac{vv}{6rn} \times \frac{2mv + 2tv - vv}{4mt}$$

+ $\frac{vv}{6rn} \times \frac{2nv - 2vv}{4mt}$, the value of an annuity for v years certain on the three joint lives (by quest. 10); and from the sum ($v100 + v100 + 2 \times v100 -$

$v100$

$$vA - \frac{vv}{6rn} \times \frac{2mv + 2tv + vv}{4mt} + \frac{vv}{6rn} \times \frac{2nv - 2vv}{4mt}$$

take $vA - vA - \frac{vv}{6rm} \times \frac{v}{2t}$, $vA - vA - \frac{vv}{6rn} \times \frac{v}{2t}$,

and $vA - vA - \frac{vv}{6rn} \times \frac{v}{2m}$, the values of annu-

ties for v years certain, for the two joint lives, whose complements are severally $m, t; n, t$; and n, m ; the remainder, viz.

$$vA + vA - \frac{vv}{6rm} \times \frac{v}{2t} - vA - \frac{vv}{6rn} \times \frac{2mv + 2tv + vv}{4mt}$$

$$+ \frac{vv}{6rn} \times \frac{2nv - 2vv}{4mt} + vA - \frac{vv}{6rn} \times \frac{v}{2t}$$

$$+ vA - \frac{vv}{6rn} \times \frac{v}{2m}, \text{ will be the value Required.}$$

But $\frac{v}{2t} + \frac{v}{2m} = \frac{2mv + 2tv}{4mt}$, whence the above

will be reduced to $vA + vA - \frac{vv}{6rm} \times \frac{v}{2t}$

$$+ vA - \frac{vv}{6rn} \times \frac{v}{4mt} + \frac{vv}{6rn} \times \frac{2nv - 2vv}{4mt} \text{ or } vA$$

$$+ vA + \frac{v}{2m} \times vA - \frac{vv}{6rm} \times \frac{v}{2t} + \frac{vv}{6rn} \times \frac{2nv - 3vv}{4mt};$$

That is vA

$$+ vA + \frac{v}{2m} \times vA - \frac{vv}{6rm} + \frac{vv}{6rn} \times \frac{2n - 3v}{2m} \times \frac{v}{2t}.$$

QUESTION XVI.

It is required to find the sum of n terms of the series

$$\frac{2m-1}{2m} \times \frac{2t-1}{2t} + \frac{2m-3}{2m} \times \frac{2t-3}{2t} + \frac{2m-5}{2m} \times \frac{2t-5}{2t}$$

&c. ?

SOLU.

44 MATHEMATICAL

SOLUTION.

By quest. 106. vol. 2. the sum of the series $\frac{2m-1}{2m} + \frac{2m-3}{2m} + \frac{2m-5}{2m}$ (n) is $1 - \frac{n}{2m} \times n$, and the sum of $\frac{2t-1}{2t} + \frac{2t-3}{2t} + \frac{2t-5}{2t}$ (n) is $1 - \frac{n}{2t} \times n$; their common differences being severally $\frac{1}{m}$, and $\frac{1}{t}$; therefore (by quest. 21. vol. 2.) the sum of the required series will be

$$\left(1 - \frac{n}{2m} \times n \right) \times \left(1 - \frac{n}{2t} \times n \right) + \frac{n+1 \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{m} \times \frac{1}{t}$$

or) $1 - \frac{n}{2m} \times n \times 1 - \frac{n}{2t} \times n + \frac{n+1 \cdot n-1}{3 \cdot 4mt}$

That is $1 - \frac{n}{2m} \times n \times 1 - \frac{n}{2t} \times n + \frac{n+1 \cdot n-1}{3 \cdot 4mt} \times n$.

But $1 - \frac{n}{2m} \times n \times 1 - \frac{n}{2t} \times n = 1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{4mt}$;

And (if we write $\frac{nn}{3 \cdot 4mt}$ for $\frac{n+1 \cdot n-1}{3 \cdot 4mt}$) then the above expression will become

$$1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{4mt} + \frac{nn}{3 \cdot 4mt} \times n$$

Now $\frac{nn}{4mt} + \frac{nn}{3 \cdot 4mt} = \left(\frac{1}{4} + \frac{1}{12} \times \frac{nn}{mt} \right) = \frac{1}{3} \times \frac{nn}{mt}$

And, therefore, the sum required will be

$$1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{3 \cdot mt} \times n$$

QUESTION

QUESTION XVII.

It is required to find the sum of v terms of the series

$$\frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} + \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t}$$

 &c. &c.

SOLUTION.

By quest. 106. vol. 2. the sum of v terms of the several component arithmetical progressions are as follows;

$$\left. \begin{aligned} \frac{2t-1}{2t} + \frac{2t-3}{2t} (v) &= 1 - \frac{v}{2t} \times v; \\ \frac{2m-1}{2} + \frac{2m-3}{2m} (v) &= 1 - \frac{v}{2m} \times v; \\ \frac{2n-1}{2n} + \frac{2n-3}{2n} (v) &= 1 - \frac{v}{2n} \times v; \end{aligned} \right\} \text{their common differences being } \left\{ \begin{aligned} \frac{1}{t}; \\ \frac{1}{m}; \\ \frac{1}{n}. \end{aligned} \right.$$

Therefore by quest. 22. vol. 2. their sum will be

$$\left(\frac{1 - \frac{v}{2t} \times v \times 1 - \frac{v}{2m} \times v \times 1 - \frac{v}{2n} \times v}{v} \right) +$$

$$\begin{aligned}
 & \frac{v+1}{2 \cdot 2 \cdot 3} \cdot \frac{v-1}{2f} \times \frac{2f-1}{2f} \times \frac{1}{m} \times \frac{1}{n} + \frac{2m-1}{2m} \times \frac{1}{f} \times \frac{1}{n} + \frac{2g-1}{2g} \times \frac{1}{f} \times \frac{1}{m} \\
 & - \frac{v+1}{2 \cdot 2 \cdot 2} \cdot \frac{v-1}{2f} \times \frac{1}{m} \times \frac{1}{n} \quad \text{(or)} \\
 & \frac{v}{2f} \times 1 - \frac{v}{2m} \times 1 - \frac{v}{2n} \times v + \frac{v+1}{2 \cdot 2 \cdot 3} \times \frac{v+1}{2 \cdot 2 \cdot 3} \times \frac{2 \times f + m + n - 3}{2fmn} \times \frac{v+1}{2 \cdot 2 \cdot 2} \times \frac{v-1}{2fmn} ;
 \end{aligned}$$

That is (by writing vw for $v+1 \times v-1$)

$$1 - \frac{v}{2f} \times 1 - \frac{v}{2m} \times 1 - \frac{v}{2n} \times v + \frac{v^3}{3} \times \frac{2 \times f + m + n - 3}{8fmn} - \frac{v^3 \times v - 1}{8fmn}$$

But $1 - \frac{v}{2f} \times 1 - \frac{v}{2m} \times 1 - \frac{v}{2n} \times v$ will, if multiplied, become

$$v^3 - \frac{v^3}{2f} - \frac{v^3}{2m} - \frac{v^3}{2n} + \frac{v^3 v}{4fn} + \frac{v^3 v}{4mn} - \frac{v^4}{8fmn} ;$$

And $\frac{v^3}{3} \times \frac{2f+2m+2n-3}{8fmn}$ will become $\left(\frac{2fv^3}{3 \cdot 8fmn} + \frac{2mv^3}{3 \cdot 8fmn} + \frac{2nv^3}{3 \cdot 8fmn} \right)$

$$- \frac{3v^3}{3 \cdot 8fmn} \text{ or } \frac{v^3}{3 \cdot 4mn} + \frac{v^3}{3 \cdot 4fn} - \frac{v^3}{8 \cdot fmn} ;$$

Also

$$\text{Also } \frac{v^3 \times v - 1}{8tmn} = -\frac{v^4}{8tmn} + \frac{v^3}{8tmn}$$

Let now these be properly ranged under each other and added as below

$$\begin{array}{r} \frac{v^4}{2t} - \frac{v^2}{2n} + \frac{v^3}{2n} - \frac{v^2}{2n} + \frac{v^3}{4tn} + \frac{v^3}{4mn} + \frac{v^3}{3 \cdot 4tn} + \frac{v^3}{3 \cdot 4mn} - \frac{v^3}{8tmn} \\ + \frac{v^3}{3 \cdot 4tn} + \frac{v^3}{3 \cdot 4mn} + \frac{v^3}{8tmn} \end{array}$$

$$\text{The sum } \left(\frac{v^4}{2t} - \frac{v^2}{2n} + \frac{v^3}{2n} + \frac{v^3}{3tn} + \frac{v^3}{3mn} - \frac{v^4}{4tmn} \right)$$

$$\text{Or } 1 - \frac{v}{2t} - \frac{v}{2n} + \frac{v^2}{3tn} + \frac{v^2}{3mn} - \frac{v^3}{4tmn} \times v \text{ will be the value required.}$$

QUESTION

QUESTION XVIII.

The complement, n , of a life being known; to find the probability of that life's failing, in any like portion of time, during the possibility of its existence.

SOLUTION.

Since the probability of the given life's continuing the first year is $\frac{n-1}{n}$; therefore the probability of its failing in that time will be $\left(1 - \frac{n-1}{n} = \frac{n-n+1}{n} =\right) \frac{1}{n}$.

When the first year is expired, then the complement of the life becomes $n-1$; the probability of its continuance another year will be $\left(\frac{n-1-1}{n-1} =\right) \frac{n-2}{n-1}$; and that of its failing in that year

$\left(1 - \frac{n-2}{n-1} = \frac{n-1-n+2}{n-1} =\right) \frac{1}{n-1}$. But if this probability is to be found at the beginning of the first year, then $\frac{n-1}{n}$ the probability of the life's continuing the first

year must be multiplied into $\frac{1}{n-1}$ the above found probability; because the life's failing, in the second year, depends on its having survived the first; and therefore

$\left(\frac{n-1}{n} \times \frac{1}{n-1} =\right) \frac{1}{n}$ will be the probability of the life's failing in the second year.

In like manner $\left(\frac{n-2}{n} \times \frac{1}{n-2} =\right) \frac{1}{n}$ will be the probability of its failing in the third year, &c.

By reasoning in the same manner, the probability of the given life's failing, in any half year, will be $\frac{1}{2n}$;

in any month, $\frac{1}{12n}$; or in any day $\frac{1}{365n}$.

QUESTION

QUESTION XIX.

The probability p , that the lives of two persons, of different ages, will be both extinct, in a given space of time, being known; to find the probability that either of them will, within that space of time, die before the other?

SOLUTION.

Let the complement of the younger life be m ; and that of the elder, n .

Since the probability of the younger life's failing, in any year, month, or day, is $\frac{1}{m}$, $\frac{1}{12m}$, or $\frac{1}{365m}$; and that of the elder life's failing, in the like times, is $\frac{1}{n}$, $\frac{1}{12n}$, or $\frac{1}{365n}$, by quest. 18. Therefore, in any such interval, the probability of the failing of the younger life, is to the probability of the failing of the elder life; as $\frac{1}{m}$, to $\frac{1}{n}$; or, as n to m .

Let now the probability of the younger life's failing first, be denoted by y ; and that of the elder, by x .

Then, as $n : m :: y : x$; Therefore $x = \frac{ny}{n}$, and $y = \frac{mx}{m}$.

But $x + y = p$, by the question.

Therefore $x + \frac{nx}{m} = p$; and $y + \frac{my}{n} = p$;

And $mx + nx = mp$; and $ny + my = np$;

Whence $x = \frac{mp}{m+n}$; and $y = \frac{np}{n+m}$.

COROL.

If the two lives are of equal ages, then $m = n$: and x
 $= \left(\frac{np}{n+n} = \frac{np}{2n} = \right) \frac{p}{2}$; also $y = \left(\frac{np}{n+n} = \right) \frac{p}{2}$.

QUESTION XX.

There is an estate of 1*l.* per annum, usually let on a lease for two lives; of which one is dropped: what fine ought to be paid to the lessor, for filling up the lease?

O R,

A is possessed of an annuity (secured by land) which, upon his decease, is to descend to *B*, if he be then living, for his life only; what is the value of *B*'s interest in that annuity?

O R,

What is the value of the reversion of an annuity (secured by land) for one life, after one?

CASE I.

When the expectant is elder than the possessor.

Then (putting n and m for the complements, and \mathcal{P} and \mathcal{M} for the values of the lives of the expectant and possessor) by arguing as in quest. 94. vol. 2.

If from the value of the expectant's life

 \mathcal{P} ,

We take the value of the joint lives of both possessor and expectant

The remainder will be the value of the reversion

$$\begin{array}{r} \mathcal{P} - \mathcal{P} - \frac{n}{6r} \times \frac{n}{2m} \\ + \mathcal{P} - \frac{n}{6r} \times \frac{n}{2m} \\ \hline \text{R.} \end{array}$$

EXAMPLE I.

If the expectant be 66, and the possessor 54, then (by exam. 3, quest. 4.) $\mathfrak{P} - \frac{n}{6r} \times \frac{n}{2m} = 1,397$ is the value required.

EXAMPLE II.

If the expectant be 66, and the possessor 43: then (by exam. 2. quest. 4.) 1,039 will be the value required.

EXAMPLE III.

If the expectant be 54 and the possessor 43: then (by exam. 1. quest. 4.) 2,095 will be the value required.

CASE II.

When the possessor is elder than the expectant.

Then put m , n , for the complements, and \mathfrak{M} , \mathfrak{P} , for the values of the lives of the expectant and possessor; and

From the value of the expectant's life } \mathfrak{M} ,

Take the value of the joint lives } $\mathfrak{P} - \mathfrak{P} - \frac{n}{6r} \times \frac{n}{2m}$,

And the remainder, } $\mathfrak{M} - \mathfrak{P} + \mathfrak{P} - \frac{n}{6r} \times \frac{n}{2m}$,

will be the value of the reversion required.

EXAMPLE I.

If the expectant be 54, and the possessor 66.

Then $\mathfrak{M} = 80,757$; $\mathfrak{P} = 7,673$; and $\mathfrak{P} - \frac{n}{6r} \times \frac{n}{2m} = 1,397$

D 2

There,

52 MATHEMATICAL

Therefore $(10,757 - 7,673 + 1,397 =) 4,481$ will be the value required.

EXAMPLE II.

If the expectant be 43, and the possessor 66.

Then $M = 12,920$; $P = 7,673$; and $P - \frac{n}{6r} \times \frac{n}{2m} = 1,039$;

Therefore $(12,920 - 7,673 + 1,039 =) 6,286$ will be the value required.

EXAMPLE III.

If the expectant be 43, and the possessor 54. then M

$= 12,920$; $P = 10,757$; and $P - \frac{n}{6r} \times \frac{n}{2m} = 2,095$;

Therefore $(12,920 - 10,757 + 2,095 =) 4,258$ will be the value required.

CASE III.

If the expectant and possessor be of the same age.

Then (by writing n for m in the first case) we shall have

$\left(P - \frac{n}{6r} \times \frac{n}{2n} \text{ or } \right) P - \frac{n}{6r} \times \frac{1}{2}$ for the value required.

EXAMPLE.

If both expectant and possessor be aged 66;

Then $P - \frac{n}{6r} = 4,468$ (by exam. 2. quest. 4.) and

$\left(\frac{4,468}{2} = \right) 2,234$ will be the value required.

QUES.

QUESTION XXI.

The respective ages of two persons, *B* the elder, and *A* the younger, being given; it is required to find the probability, that *B* has of surviving *A*.

SOLUTION.

Let the complements of the lives of *B* and *A*, be (severally) denoted by n and m : then the terms of the series $\frac{2n-1}{2n}$, $\frac{2n-3}{2n}$, $\frac{2n-5}{2n}$, &c. will be the several expectations of the life of *B*, for the first, second, third, &c. years; or the probabilities of the continuance of that life, to the expiration of the whole, or at least of the half of each of those years (by quest. 105. vol. 2.) Also $\frac{1}{m}$ will be the probability of the failing of the life of *A*, in the first year; that is, of the decease of *A*, either in the first, or in the second, half of that year.

Now, if the expectation of *B*'s life, for the first year (*viz.* $\frac{2n-1}{2n}$) be multiplied by $\frac{1}{m}$, the probability of *A*'s dying in that year; then the product, $\frac{2n-1}{2n} \times \frac{1}{m}$, will be the probability of the survivorship's taking place in that year: for if *A* be supposed to die in the first half year, then the expectation, $\frac{2n-1}{2n}$, includes the probability of *B*'s living, at least, to the end of that half year, and consequently of his surviving *A* for that time; and if *A* be supposed not to die until the second half year, then the same expectation includes the probability of *B*'s living to the end of the year, and (by that means) of his surviving *A*, for that time also.

If it were certain that the survivorship would take place in the first year, then this calculation would proceed no

farther; but, since both the persons may outlive the first year, we must proceed to find the probability of its taking place in the second: now the expectation of B 's life, for the second year, is $\frac{2n-3}{2n}$ and the probability that A , having survived the first year, shall die in the second is (by quest. 18.) $\frac{1}{m}$; consequently their product,

$\frac{2n-3}{2n} \times \frac{1}{m}$, will (for the same reasons as before) be the probability of the survivorship's taking place in the second year.

And, if we continue to argue in the same manner, it will appear that $\frac{2n-5}{2n} \times \frac{1}{m}$, $\frac{2n-7}{2n} \times \frac{1}{m}$, &c. will be the probabilities of the survivorship's taking place, in the third, fourth, &c. years; and consequently, that

$\left(\frac{2n-1}{2n} \times \frac{1}{m} + \frac{2n-3}{2n} \times \frac{1}{m} + \frac{2n-5}{2n} \times \frac{1}{m} (n) \text{ or } \right)$
 $\frac{1}{m} \times \frac{2n-1}{2n} + \frac{2n-3}{2n} + \frac{2n-5}{2n} (n)$ will be the whole probability of B 's surviving A .

But since (by quest. 105. vol. 2.) the sum of the series $\frac{2n-1}{2n} + \frac{2n-3}{2n} + \frac{2n-5}{2n} (n)$ is $\frac{n}{2}$; and, since every term of that series is above multiplied by the constant factor, $\frac{1}{m}$; it follows, that $\left(\frac{n}{2} \times \frac{1}{m} = \right) \frac{n}{2m}$ will be the whole probability of B 's surviving A .

QUESTION XXII.

The same lives being proposed, as in quest. 21. it is required to find the probability, that A , the younger, has of surviving B , the elder?

If

If we assume m and n for their complements of life, then (arguing as before) the terms of the series $\frac{2m-1}{2m}$,

$\frac{2m-3}{2m}$, $\frac{2m-5}{2m}$, &c. which express the yearly expectations

of A 's life, are to be severally multiplied by $\frac{1}{n}$, the yearly probability of B 's dying: but since that probability does (by the hypothesis) become a certainty, at the expiration of n years; it follows, that n terms only of the series resulting (viz. $\frac{2m-1}{2mn} + \frac{2m-3}{2mn} + \frac{2m-5}{2mn}$) will exhibit the value required.

Now, (by quest. 106. vol. 2.) the sum of n terms of the series, $\frac{2m-1}{2m} + \frac{2m-3}{2m} + \frac{2m-5}{2m}$ &c. is

$1 - \frac{n}{2m} \times n$; which sum, being multiplied by the constant factor, $\frac{1}{n}$, produces $1 - \frac{n}{2m}$ for the probability required.

SCHOLIUM.

Here it may be remarked, that the sum of the two probabilities of survivorship above found, viz. $\frac{n}{2m}$,

and $1 - \frac{n}{2m}$, is unity; for it may be esteemed a certainty, that one of the two persons will survive the other.

COROLL.

Hence, if the two lives proposed be of equal ages, then their probabilities of survivorship will be equal, viz.

$$\left(\frac{n}{2n} \text{ or } \right) \frac{1}{2}$$

D 4

Q U E S.

QUESTION XXIII.

A has left a legacy of (or an estate which will sell for) \mathcal{P} pounds, to B , if he be alive at the time of A 's decease; but if not, then the heirs or assigns of B are not to receive the benefit thereof; what is the present worth of B 's interest, in that legacy or estate?

CASE I.

If the expectant be elder than the possessor;

Let x be the complement of the expectant's life, and m that of the possessor; then (by quest. 21.) the probability of his receiving the legacy in the first, second, third, &c. years, will be, $\frac{2x-1}{2nm}$, $\frac{2x-3}{2nm}$, $\frac{2x-5}{2nm}$, &c. and con-

sequently, the present worths of the sum \mathcal{P} , so to be received, will be $\frac{2x-1}{2nmr} \mathcal{P}$, $\frac{2x-3}{2nmr^2} \mathcal{P}$, $\frac{2x-5}{2nmr^3} \mathcal{P}$, &c. in

each of which, $\frac{\mathcal{P}}{m}$, is a constant factor; and therefore

$$\left(\frac{\mathcal{P}}{m} \times \frac{2x-1}{2nr} + \frac{2x-3}{2mr^2} + \frac{2x-5}{2mr^3} (n) = \right) \frac{\mathcal{P}n}{m}, \text{ will}$$

be the present worth of B 's interest in the estate.

EXAMPLE I.

Suppose B , aged 66, is to receive a legacy of twenty five pounds, or an estate in fee simple of \mathcal{L} per ann. if he survives A , who is 54 years of age: what is that expectation worth, allowing compound interest at 4 per cent?

Here $\mathcal{P} = 25$; $n = 7,673$; and $m = 32$;

Th. $\left(\frac{7,673 \times 25}{32} = \right) 5,995\mathcal{L}$. will be the value required.

Note,

Note, The symbol \mathcal{P} , in the above solution mentioned; denotes the legacy, or the present worth of the estate in fee simple (of what annual rent soever) supposed to devolve to the expectant on the death of the possessor: whereas P , the symbol (in this work) used before this question, denotes the present worth of an estate of only 1£ per ann. Therefore these two values will coincide only in such a case, as is proposed in the above example.

EXAMPLE II.

If B , aged 66, is to receive the aforesaid legacy, or estate, if he survives A , aged 43.

Then $\mathcal{P} = 25$; $\mathcal{P} = 7,673$; and $m = (86 - 43 =) 43$.

Whence $\left(\frac{7,673 \times 25}{43} = \right) 4,461\text{£}$. the value required.

EXAMPLE III.

If B , aged 54, is to receive the same, if he survives A , aged 43.

Then $\mathcal{P} = 25$; $\mathcal{P} = 10,757$; and $m = 43$:

Whence $\left(\frac{10,757 \times 25}{43} = \right) 6,254\text{£}$. is the value required.

CASE II.

If the expectant be younger than the possessor; let n be the complement of the possessor's life, and m that of the expectant.

Then, by comparing together the arguments in quest. 22. and in the first case of this, it will appear, that

$\frac{\mathcal{P}}{n} \times \frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3}(n)$, will be the present value required.

But $\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3}(n)$ is (by quest. 2.)

$D \ 5$ equal

equal to $(P - \frac{P}{m} \times \frac{m-1}{r^n} - \frac{1}{a} \times \frac{1}{r^n} - \frac{1}{r^n})$

Or) \therefore Therefore $\frac{P}{r^n} \times \frac{1}{r^n}$ will be the value required.

EXAMPLE I.

Suppose *B*, aged 54, is to receive a legacy of 25£. or an estate is free simple of 1£. per ann. if he survives *A*, who is 66 years of age; what is that expectation worth, allowing compound interest at 4 per cent?

Here $P = 25$; $\frac{1}{r^n} = 9,891$ (by quest. 2.) and $n = 205$.

Therefore $(\frac{25}{20} \times 9,891 =) 12,364$ will be the value required.

EXAMPLE II.

Let *B*, aged 43, be to receive the same legacy or estate if he survives *A*, aged 66.

Here $P = 25$; $\frac{1}{r^n} = 10,838$; and $n = 20$;

Th. $(\frac{10,838 \times 25}{20} =) 13,547\text{£.}$ will be the value required.

EXAMPLE III.

If *B*, aged 43, is to receive the same, if he survives *A*, aged 54.

Here $P = 25$; $\frac{1}{r^n} = 12,578$; and $n = 32$

Th. $(\frac{12,578 \times 25}{32} =) 9,827\text{£.}$ will be the value required.

SCHOLIUM.

By question 22. the sum of the probabilities of the younger life's surviving the elder; *viz.* the sum of the series $\frac{2m-1}{2mn} + \frac{2m-3}{2mn} + \frac{2m-5}{2mn} (n)$ appeared to be $1 - \frac{n}{2m}$; and (by case 2. of this) the sum of the series $\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3} (n)$ is $\frac{n}{2m}$: therefore, in all cases, wherein the sum of the probabilities of survivorship is denoted by the expression, $1 - \frac{n}{2m}$, or one similar thereto, we may write, $\frac{n}{2m}$, for the present value of 1L. dependent on that probability.

CASE III.

If the possessor and expectant are of equal ages.

Then (putting n for their common complement) the value required will be $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} + \frac{2n-5}{2nr^3} (n) \times \frac{P}{n}$

Or $\frac{P}{n}$.

EXAMPLE.

Suppose both possessor and expectant, to be aged 66, the estate and rate of interest being as before.

Then $P = 7,673$; $P = 25$; and $n = 20$;

Whence $\frac{7,673 \times 25}{20} = 9,591$ will be the value required.

C O R O L.

If the expectant should, during the life of the possessor, want to borrow a sum of money, on his interest in the estate or legacy, dependent on the above survivorship; then the utmost that can be advanced to him thereon, will be the above found value of his interest therein, in which case he sells his whole expectation; but if he borrows a lesser sum, then the 25th question naturally arises.

S C H O L I U M.

The reader will observe, that there is a very considerable difference between the two reversions, or survivorships, proposed in questions 20 and 23; in the former the expectant is (for the time which he survives the possessor) to receive divers annual sums; whereas, in the latter, he is to receive on the death of the possessor only one gross sum or legacy: or thus, in the former case, the expectant is (at the death of the possessor) to enjoy an annuity for the remainder of his life, which annuity will be of different values, according to the age at which he shall happen to come into possession; whereas in the latter, (if he survives the possessor) he and his heirs are to enjoy an annuity for ever; which perpetuity will, therefore, be of the same value, at any time, when it may come into his possession.

For this reason, therefore, they are calculated in manners so different; viz. the former case depends on the solution of (quest. 94. vol. 2.) wherein the probabilities of the possessor's dying, before the expiration of one, two, 3, &c. years, are severally multiplied into the probabilities of the expectant's living, to the end of those years; and therefore the probabilities of receiving a payment of the annuity in the second year, doth not depend on the not having received it in the first: but in the second case the probability of the possessor's dying in any one year, is constantly multiplied into the other's expectation of life, for the first, second, third, &c. years; which constant probability will (by quest. 18.) appear, in the second year,

to suppose the possessor's having survived the first; and consequently, the probability of receiving the legacy, in the second year, does (in this calculation) entirely depend on the not having received it in the first.

This may serve to account for a similar difference in the manner of calculating questions of those kinds, in the ensuing parts of this work.

QUESTION XXIV.

Things being as in the last question, suppose that *A* has (in case *B* dies before him) left the legacy, or estate, (worth \mathcal{P} pounds) to *C* and his heirs; what is the present worth of their interest in that reversion?

SOLUTION.

If *B* had no interest in the legacy, or estate, then *C*, or his heirs, would be entitled to the same, immediately on the death of *A*: if, therefore, from the value of the reversion of the legacy or estate, after the life of *A*, there be taken the interest of *B* therein; the remainder will be the value of the expectation of *C*, or his heirs.

CASE I.

If *B*, the expectant, be elder than *A*, the possessor. Let n , m , represent the complements of their lives; and \mathcal{A} , \mathcal{B} , the values of annuities for them. Then (by question 89. vol. 2.) the present value of \mathcal{A} to be received on the death of *A* will be $1 - r - 1 \times \mathcal{A}$; whence the value of the legacy, or estate in question will be $1 - r - 1 \times \mathcal{A} \times \mathcal{P}$; and (by case 1. question 23.) the present value of *B*'s interest therein is $\frac{\mathcal{B} \mathcal{A}}{m}$; therefore the present value of the interest of *C*, and his heirs, therein will be

$$1 - r$$

$$1 - r - 1 \times \frac{P}{m} \times P - \frac{P^2}{m} \text{ or } 1 - r - 1 \times \frac{P}{m} - \frac{P}{m} \times P.$$

EXAMPLE.

Suppose *A* and *B* to be aged 54 and 66; interest at 4 per cent. and the legacy 25£.

Then $\frac{P}{m} = 10,757$; $\frac{P^2}{m} = 7,673$; $m = 32$; $r = 1 = 0,04$; and $P = 25$: now $10,757 \times 0,04 = ,43028$;
 $\frac{7,673}{32} = ,240$;

And $1 - 0,430 - 0,240 = 0,330$; theref. $(25 \times 0,330 =) 8,25$ will be the value required.

CASE II.

If *B*, the expectant, be younger than *A*, the possessor.

Let *m* represent the complement of the expectant, and *n* that of the possessor.

Also let $\frac{P}{m}$ represent the value of an annuity for the possessor's life, and $\frac{P}{n}$ that of an annuity for *n* years, if the expectant lives so long.

Then (by quest. 89. vol. 2.) the value of $\frac{P}{m}$ pound to be received at the death of the possessor, will be

$1 - r - 1 \times \frac{P}{m} \times \frac{P}{m}$; and (by case 2. quest. 23.) the value of *B*'s interest in the legacy or estate, will be

$\frac{P}{n} \times \frac{P}{m}$; therefore $(1 - r - 1 \times \frac{P}{m} \times \frac{P}{m} - \frac{P}{n} \times \frac{P}{m})$ or

$1 - r - 1 \times \frac{P}{m} - \frac{P}{n} \times \frac{P}{m}$ will be the value required.

EXAMPLE.

Suppose *A* and *B*, to be aged 66 and 54; interest at 4 per cent, and the legacy 25£.

Then $\frac{P}{m} = 7,673$; $\frac{P}{n} = 9,891$; $n = 20$; $r = 1 = 0,04$; and $P = 25$.

Now

Now $7,673 \times 0,04 = 30692$; $\frac{9,891}{20} = 0,4995$;

And $1 - 0,307 - 0,500 = 0,193$; Therefore $(25 \times 0,193 =) 4,825$ will be the value required.

CASE III.

If the expectant and possessor be of the same age.

Then (putting n for the complement and P for the va-

lue of a life of that age) $1 - r - 1 \times P - \frac{P}{n} \times P$ will be the value required.

EXAMPLE.

Suppose A and B to be each 66 years of age; interest at 4. per cent. and the legacy 25 pounds.

Then $P = 7,673$; $n = 20$; $r = 1 = 0,04$; and $P = 25$ £.

Then $7,673 \times 0,04 = 30692$; $\frac{7,673}{20} = 0,38365$;

And $1 - 0,307 - 0,384 = 0,309$; Th. $(0,309 \times 25 =) 7,725$ will be the value required.

As the values of the reversions of an estate or legacy, dependent on the failure of any of the survivors hereafter calculated, may be obtained, by arguing in the same manner as in this question, no more instances thereof are inserted in this work.

QUESTION XXV.

B (who is entitled to a legacy of or an estate worth P pounds, if he survives the death of A) would borrow a sum of money of C upon the credit thereof; on condition that if he, B , dies before A , then C shall entirely lose his money: what sum ought C to receive on the death of A , if B be then living, for every pound now lent?

SOLU.

SOLUTION.

Since C is to lose the sum lent, unless B survives A , it is plain, that he has purchased the expectation of a sum of money, dependent on that survivorship; the present values of which, in the three possible cases, were computed in question XXIII; if, therefore, equations be severally made between unity, or one pound the present worth given in the question, and the results of those cases; then the values of P , (the sum receivable when the survivorship takes place) being found, by reducing those equations, will be the answers.

CASE I.

If the expectant be elder than the possessor.

Let m be the complement of the possessor, and P the value of an annuity (secured by land) for the expectant's life.

Then $1 = \frac{P \cdot P}{m}$; whence $\frac{m}{P} = P$.

EXAMPLE I.

What sum ought C to receive at the death of A , aged 54, for every pound lent, on the contingency that B , (who is 66) shall survive A ?

Here $m = 32$; and $P = 7,673$; therefore $P = \left(\frac{32}{7,673} = \right) 4,171$ will be the sum required.

EXAMPLE II.

If the expectant be 66, and the possessor 43 years old.

Then $m = 43$; and $P = 7,673$; therefore $P = \left(\frac{43}{7,673} = \right) 5,604$ will be the sum required.

EX-

EXAMPLE III.

If the expectant be 54, and the possessor 43 years old.
Then $m = (86 - 43) = 43$; and $\mathcal{M} = 10,757$;
therefore $\mathcal{P} = \left(\frac{43}{10,757} = \right) 3,998$ will be the sum
required.

CASE II.

If the expectant be younger than the possessor:
Let n be the complement of the possessor's life, and
 \mathcal{M} be the value of an annuity (secured by land) for n
years certain, if the expectant live so long, found by
quest. 2.

Then $1 = \frac{\mathcal{P}}{n} \times \mathcal{M}$; whence $\frac{n}{\mathcal{M}} = \mathcal{P}$.

EXAMPLE I.

If the expectant be 54, and the possessor 66 years of
age. then $n = 20$; and $\mathcal{M} = 9,891$; therefore
 $\left(\frac{20}{9,891} = \right) 2,022$ will be the sum required.

EXAMPLE II.

If the expectant be 43 and the possessor 66 years of
age.

Here $n = 20$; and $\mathcal{M} = 10,838$; therefore
 $\left(\frac{20}{10,838} = \right) 1,846$ will be the sum required.

EXAMPLE III.

If the expectant be 43 and the possessor 54 years of
age.

Here $n = 32$; and $\mathcal{M} = 12,578$; therefore
 $\left(\frac{32}{12,578} = \right) 2,544$ will be the sum required.

CASE

66 MATHEMATICAL

CASE III.

If the expectant and possessor are of the same age.

Let n be the complement and P the value of an annuity (secured by land) for a life of that age.

Then $1 = \frac{nP}{n}$; whence $\frac{n}{P} = P$.

EXAMPLE.

If the persons are each aged 66.

Then $n = 20$; and $P = 7,673$; Therefore $\frac{20}{7,673} = 2,607$ will be the sum required.

The same method of proceeding will give the sum of money which ought to be received for £ . lent; when the repayment depends on any other kind of survivorship.

QUESTION XXVI.

Things being as in the last question, if (instead of receiving a sum of money on the decease of A) C should chuse an annuity for the remainder of B 's life, to commence at the decease of A ; it is required to determine how much per annum he should receive for every pound now lent.

SOLUTION.

Here it is evident that C purchases the reversion of an annuity for B 's life after the decease of A ; the values of which in all the possible cases will (if x be put to represent the annual payment) be found by question 20: If therefore equations be severally made between unity, or 1 pound, the given present worth of such annuity; and the results of those cases; then the values of x being found, by reducing those equations, will be the answers.

CASE

CASE I.

When the expectant is elder than the possessor.

Then $\frac{n}{6r} \times \frac{n}{2m} x = 1$; therefore $x =$

$$\frac{2m}{\frac{n}{6r} \times n}.$$

EXAMPLE.

If the expectant be 66, and the possessor 54.

Then the reversion of an annuity of 1*l.* is (per ex. 1. case 1. quest. 20) 1,397; and $\left(\frac{1}{1,397} =\right)$ 0,716 will be the annual payment required.

CASE II.

When the possessor is elder than the expectant:

Then $\frac{n}{6r} + \frac{n}{2m} x = 1$; therefore $x =$

$$\frac{1}{\frac{n}{6r} + \frac{n}{2m}}.$$

EXAMPLE.

If the expectant be 54, and the possessor 66.

Then $\left(\frac{1}{4,481} =\right)$ 0,223 will be the annual payment required.

68 MATHEMATICAL

CASE III.

When the possessor and expectant are of the same age.

$$\text{Then } 1 - \frac{n}{6r} \times \frac{1}{2} x = 1; \text{ Th. } x = \frac{2}{1 - \frac{n}{6r}}.$$

EXAMPLE.

If both expectant and possessor be aged 66.

Then $\left(\frac{1}{2,234}\right) 0,448$ will be the annual payment required.

The same method of proceeding will give the annual payment, which ought to be made in consequence of the lending of 1*£*. when the commencement of such payments depends on any other kind of reversion.

QUESTION XXVII.

B (who will become possessed for life of a considerable estate, if he survives *A*, the present possessor) would borrow of *C* a sum of money upon the credit thereof: now it is agreed between them, that *C* shall be repaid the sum (so borrowed) by an annuity (secured on the land) for his (*C*'s) life, to commence on the decease of *A* (if *B* be then alive) and to continue as long as *B* shall possess the estate; that is, as long as *B* shall live: what sum ought *C* to lend, in consideration of the reversion of an annuity of 1*£*. so circumstanced?

O R,

What is the value of the reversion of an annuity (secured by land) for the joint lives of *B* and *C*, after the decease of *A*?

SOLU-

SOLUTION.

By the solution of quest. 95. vol. 2. fol. 293; if from the value of the joint lives of the two expectants, be taken the value of the three joint lives; the remainder will be the value of the reversion.

CASE I:

When the possessor is younger than the expectants.

I let t be the complement of the possessor's life; m and n , those of the expectants; and P the value of the single life, whose complement, x , is the least.

Then (by quest. 4.) $P - P - \frac{n}{6r} \times \frac{n}{2m}$ will be the value of the joint lives of the two expectants;

And (by quest. 9.) $P - P - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{4mt}$ will be the value of the three joint lives.

Therefore, $\left(P - \frac{n}{6r} \times \frac{n}{2t} - \frac{nn}{4mt} \text{ or } \right)$

$P - \frac{n}{6r} \times 1 - \frac{n}{2m} \times \frac{n}{2t}$ will be the value of the reversion required.

EXAMPLE.

If the possessor be 43, and the expectants 54 and 66: then $t = 43$; $m = 32$; $n = 20$; $P = 7,673$; and

$$\frac{n}{6r} = 3,205; \text{ also } \frac{n}{2m} = \left(\frac{20}{2 \times 32} = \right) \frac{5}{16}$$

Now from $P = 7,673$; and from $1 = \frac{16}{16}$;

Take $\frac{n}{6r} = 3,205$; take $\frac{n}{2m} = \frac{5}{16}$;

Remains

4,468:

Remains

$\frac{11}{16}$.

Then

70 - MATHEMATICAL

Then $(4,468 \times \frac{11}{16} \times \frac{20}{2.43} =)$ 0,714 will be the value required.

C O R O L. I.

If the expectants are of equal ages; then $m = n$; whence $1 - \frac{n}{2m} = (1 - \frac{1}{2} =) \frac{1}{2}$; and the value of the reversion will become $\mathcal{R} - \frac{n}{6r} \times \frac{n}{4t}$.

C O R O L. II.

If the three persons are of equal ages; then $t = m = n$; and the value of the reversion will become $\mathcal{R} - \frac{n}{6r} \times \frac{1}{4}$.

C A S E II.

When the expectants are one elder, and the other younger than the possessor.

Let m be the complement of the possessor's life; t and n those of the expectants; and \mathcal{R} the value of the single life, whose complement n is the least.

Then from, $\mathcal{R} - \frac{n}{6r} \times \frac{n}{2t}$, the joint lives of the expectants;

Take, $\mathcal{R} - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{4mt}$,
the three joint lives;

And the remainder $(\mathcal{R} - \frac{n}{6r} \times \frac{n}{2m} - \frac{nn}{4mt} \text{ or })$

$\mathcal{R} - \frac{n}{6r} \times 1 - \frac{n}{2t} \times \frac{n}{2m}$ will be the value of the reversion required.

E X A M P L E.

REPOSITORY.

57

EXAMPLE.

If the possessor be 54; and the expectants 43 and 66,
then $t = 43$; $m = 32$; $n = 20$; and $\mathcal{P} - \frac{n}{6r} =$
4,468 (as in the example to case 1.); $\frac{n}{2t} = \left(\frac{20}{2 \cdot 43} = \right)$
 $\frac{10}{43}$ and $\left(1 - \frac{10}{43} = \right) \frac{33}{43}$.

Whence $\left(4,468 \times \frac{33}{43} \times \frac{20}{2 \cdot 32} = \right) 1,072$, will be
the value of the reversion required.

COROL. I.

If the younger expectant be of the same age with the
possessor; then $t = m$; and the value of the reversion

will become $\mathcal{P} - \frac{n}{6r} \times 1 - \frac{n}{2m} \times \frac{n}{2m}$.

COROL. II.

If the elder expectant be of the same age with the pos-
sessor; then $m = n$; whence the value of the re-

version will become $\mathcal{P} - \frac{n}{6r} \times 1 - \frac{n}{2t} \times \frac{1}{2}$.

CASE III.

When the possessor is elder than the expectants.

Let n be the complement of the possessor's life, t and
 m those of the possessors; \mathcal{P} and \mathcal{P} the values of the
single lives, whose complements are m and n .

Then

72 MATHEMATICAL

Then from, $\frac{m}{6r} - \frac{m}{6r} - \frac{n}{6r} \times \frac{n}{2t}$, the joint lives of the expectants;

Take, $\frac{n}{6r} - \frac{n}{6r} - \frac{n}{6r} \times \frac{n}{2m} + \frac{n}{2t} - \frac{nn}{4mt}$, the three joint lives; and the Remainder will be the answer.

But here, since there is no common factor in both expressions; the joint lives of the two expectants, and the three joint lives, must be separately computed; and their difference found by subtraction.

EXAMPLE.

If the possessor be 66, and the expectants 43 and 54.

Then (by exam. 1. quest. 4.) the joint lives of the expectants } = 8,662,

And (by quest. 9.) the three joint lives = 5,562;

Th. the value of the reversion will be 3,100.

COROLL.

If the expectants are of equal ages; then $t = m$; the joint lives of the expectants will become $\frac{m}{6r} -$

$\frac{m}{6r} - \frac{m}{6r} \times \frac{1}{2}$; and the three joint lives

$(\frac{n}{6r} - \frac{n}{6r} - \frac{n}{6r} \times \frac{n}{m} - \frac{nn}{4mm} \text{ or } \frac{n}{6r} -$

$\frac{n}{6r} - \frac{n}{6r} \times 1 - \frac{n}{4m} \times \frac{n}{m}$; but these values must (as

before) be separately computed; and their difference will be the answer.

QUESTION

QUESTION XXVIII.

There is an estate of 1*l.* per ann. usually let on a lease for three lives; of which two are already fallen: what fine ought to be paid to the lessor, for filling up the lease?

O R,

What is the value of the reversion of an annuity (secured by land) for the longest of two lives, after the decease of one?

SOLUTION.

By the solution of quest. 97. vol. 2. fol. 308; if, from the value of the longest of three lives the value of the possessor's life be taken, the remainder will be the value of the reversion.

CASE I.

When the possessor is younger than the expectants.

Let the complement of the possessor's life be denoted by t ; and those of the expectants by m and n ; and let F , M , and N be severally the values of annuities on these lives, whose complements are t , m and n .

Then, from $F + M - \frac{m}{6r} \times \frac{m}{2t} +$

$N - \frac{n}{6r} \times \frac{nn}{4mt}$, the value of the longest of the three lives (by quest. 14.); take F the value of the possessor's life; and the remainder $(M - \frac{m}{6r} \times \frac{m}{2t} +$

$N - \frac{n}{6r} \times \frac{nn}{4mt}$ or)

$M - \frac{m}{6r} \times m + N - \frac{n}{6r} \times \frac{nn}{2m} \times \frac{1}{2t}$ will be the answer.

VOL. III.

E

COROL:

COROL. I.

If the expectants are of equal ages: then $m = n$, and
 $\mathcal{R} = \mathcal{R}$; whence $\left(\mathcal{R} - \frac{n}{6r} \times n + \frac{n}{2} \times \frac{1}{2t} \text{ or } \right)$
 $\mathcal{R} - \frac{n}{6r} \times \frac{3n}{4t}$ will be the answer.

COROL. II.

If the three lives are of equal ages; then
 $\mathcal{R} - \frac{n}{6r} \times \frac{1}{4}$ will be the value of the reversion required;

CASE II.

When the expectants are one elder, and the other younger, than the possessor.

Then (changing the symbols, as in case 2. quest. 27. and proceeding as before.)

$\mathcal{R} - \mathcal{R} + \mathcal{R} - \frac{m}{6r} \times \frac{m}{2t} + \mathcal{R} - \frac{n}{6r} \times \frac{nn}{4mt}$
 will be the value of the reversion required.

COROL. I.

If the younger expectant be of the same age with the possessor; then $\mathcal{R} = \mathcal{R}$, and $t = m$; whence,

$\mathcal{R} - \frac{m}{6r} \times \frac{1}{2} + \mathcal{R} - \frac{n}{6r} \times \frac{nn}{4mm}$, will be the value required.

COROL. II.

If the elder expectant be of the same age with the possessor; then $\mathcal{R} = \mathcal{R}$, and $m = n$, whence

$\mathfrak{F} - \mathfrak{R} + \mathfrak{R} - \frac{n}{6r} \times \frac{3n}{4t}$ will be the value required.

CASE III.

When the possessor is elder than the expectants.

Then (using the symbols as in case 2. quest. 27. and proceeding as before) $\mathfrak{F} - \mathfrak{R} + \mathfrak{R} - \frac{n}{6r} \times \frac{m}{2t}$
 $+ \mathfrak{R} - \frac{n}{6r} \times \frac{nn}{4mt}$, will be the value of the reversion required.

COROL.

When the expectants are of equal ages; then $\mathfrak{F} = \mathfrak{R}$,
 and $t = m$; whence $\mathfrak{R} - \mathfrak{R} + \mathfrak{R} - \frac{n}{6r} \times \frac{1}{2}$
 $+ \mathfrak{R} - \frac{n}{6r} \times \frac{nn}{4mt}$ will be the value required.

QUESTION XXIX.

There is an estate of 1*l.* per annum, usually let on a lease for three lives; of which one is dropt; what fine ought to be paid to the lessor, for filling up the lease?

O R.

What is the value of the reversion of an annuity (secured by land) for one life, after the longest of two lives?

SOLUTION.

By the solution of quest. 101. vol. 2. fol. 318. if, from the value of the longest of the three lives, the value of the

76 MATHEMATICAL

longest of the two possessors lives be taken; the remainder will be the value of the reversion.

CASE I.

When the expectant is elder than the possessors.

Let n represent the complement of the expectant, m, t , those of the possessors; and D, M, F , the values of the single lives, whose complements are severally n, m, t .

Then, if from $F + M - \frac{m}{or} \times \frac{m}{2t} +$

$M - \frac{n}{6r} \times \frac{nn}{4mt}$, the value of the longest of the 3 lives;

we take $F + M - \frac{m}{or} \times \frac{m}{2t}$ the value of the longest of the possessor's lives; the remainder,

$D - \frac{n}{6r} \times \frac{nn}{4mt}$, will be the value of the reversion required.

EXAMPLE.

If the two possessors are of the ages 43 and 54, and the expectant 66.

Then $D - \frac{n}{6r} = 4,468$; and $(4,468 \times \frac{20 \cdot 20}{4 \cdot 32 \cdot 43} = 0,325$, will be the value of the reversion required.

COROL. I.

If the possessors are of equal ages.

Then $t = m$; and $(D - \frac{n}{6r} \times \frac{nn}{4mm} \text{ or } D - \frac{n}{6r} \times \frac{n}{2m})$ will be the value of the reversion required.

COROL.

COROL. II.

If the 3 persons are of equal ages; then $t=m=n$; and the value of the reversion will become $D - \frac{n}{or} \times \frac{1}{4}$.

CASE II.

When the possessors are, the one elder, and the other younger than the expectant.

Here m will represent the complement of the expectant, n and t those of the possessors.

Then, if from $F + D - \frac{m}{6r} \times \frac{m}{2t} + D - \frac{n}{6r} \times \frac{nn}{4mt}$;

We take $F + D - \frac{n}{or} \times \frac{n}{2t}$;

The remainder $(D - \frac{m}{or} \times \frac{m}{2t} - D - \frac{n}{6r} \times \frac{n}{2t} - \frac{nn}{4mt})$

Or) $D - \frac{m}{or} \times \frac{m}{2t} - D - \frac{n}{6r} \times 1 - \frac{n}{2m} \times \frac{n}{2t}$

will be the value of the reversion required.

EXAMPLE.

If the expectant is 54 years old; the possessors being 43 and 66.

Then $D - \frac{m}{or} = 5,629$; $D - \frac{n}{6r} = 4,468$; $\frac{n}{2m} =$

$(\frac{20}{86}) = \frac{1}{18}$; And $1 - \frac{n}{2m} = \frac{17}{18}$;

Then $(5,629 \times \frac{32}{86} - 4,468 \times \frac{17}{18} \times \frac{20}{86}) = 1,381$ will be the value required.

COROL. I.

If the elder possessor and expectant are of the same age.

Then $m=n$, and $\mathcal{P} = \mathcal{E}$; whence the value of the reversion will become

$$\mathcal{P} - \frac{n}{6r} \times \frac{n}{2t} = \mathcal{P} - \frac{n}{6r} \times 1 = \frac{n}{2n} \times \frac{n}{2t};$$

That is

$$\left(\mathcal{P} - \frac{n}{6r} \times 1 = 1 - \frac{1}{2} \times \frac{n}{2t} = \right) \mathcal{P} - \frac{n}{6r} \times \frac{n}{4t}$$

COROL. II.

If the younger possessor and expectant are of the same age.

Then $t=m$; and the value of the reversion will become,

$$(\mathcal{P} - \frac{m}{6r} \times \frac{m}{2m} = \mathcal{P} - \frac{n}{6r} \times 1 = \frac{n}{2m} \times \frac{n}{2m})$$

$$\text{Or } \mathcal{P} - \frac{m}{6r} \times \frac{1}{2} = \mathcal{P} - \frac{n}{6r} \times 1 = \frac{n}{2m} \times \frac{n}{2m};$$

CASE III

When the expectant is younger than the possessors.

Let t be the complement of the expectant; m, n , those of the Possessors; and $\mathcal{P}, \mathcal{M}, \mathcal{E}$, the values of the single lives, whose complements are t, m, n .

Then from the value of the longest of the three lives, viz.

$$\mathcal{P} + \mathcal{M} - \frac{m}{6r} \times \frac{m}{2t} + \mathcal{E} = \frac{n}{6r} \times \frac{nn}{4mt}; \text{ take the value of the longest of the two possessor's lives, viz. } \mathcal{P} + \mathcal{M}$$

$$\mathcal{M} + \mathcal{E} - \frac{n}{6r} \times \frac{n}{2m}; \text{ and the remainder}$$

$$(\mathcal{P} - \mathcal{M} + \mathcal{E} - \frac{m}{6r} \times \frac{m}{2t} = \mathcal{P} - \frac{n}{6r} \times \frac{n}{2m} = \frac{nn}{4mt})$$

REMAINDER

EE

Or

Or) $\bar{r} - \bar{r} + \bar{r} - \frac{n}{6r} \times \frac{n}{2t} - \bar{r} - \frac{n}{6r} \times 1 - \frac{n}{2t} \times \frac{n}{2n}$
 will be the value of the reversion required.

EXAMPLE.

If the two possessors are of the ages 54 and 66, and the expectant 43.

Then $\bar{r} = 12,920$; $\bar{r} = 10,757$; $\bar{r} = 7,673$; $n = 20$;

$m = 32$; $t = 43$; $\bar{r} - \frac{n}{6r} = 5,629$; $\bar{r} - \frac{n}{6r} = 4,468$;

$\frac{n}{2t} = \left(\frac{20}{86} = \right) \frac{10}{43}$; and $1 - \frac{n}{2t} = \frac{1}{2}$;

Th. $(12,920 - 10,757 + 5,629 \times \frac{1}{2} - 4,468 \times \frac{1}{2} \times \frac{20}{86} =) 3,187$ will be the value of the reversion required.

COROL.

If the possessors are of equal ages.

Then $\bar{r} = \bar{r}$, and $m = n$; whence the value of the reversion will become,

$$\bar{r} - \bar{r} + \bar{r} - \frac{n}{6r} \times \frac{n}{2t} - \bar{r} - \frac{n}{6r} \times \frac{n}{2n} - \frac{nn}{4nt};$$

That is $(\bar{r} - \bar{r} + \bar{r} - \frac{n}{6r} \times \frac{n}{2t} - \frac{1}{2} + \frac{n}{4t} =)$

$$\bar{r} - \bar{r} + \bar{r} - \frac{n}{6r} \times \frac{n}{4t} - \frac{1}{2}$$

QUESTION XXX.

A will be entitled to an annuity (secured by land) for the remainder of his life, after the decease of B; if B survives C, the present possessor thereof; but if B dies be-

fore C , then the annuity will (upon C 's decease) descend to D , a person of the same age with A ; the respective interests of A and D , in that annuity, are required.

SOLUTION.

Since A and D are of equal ages; they are, between them, entitled to the reversion of an annuity, for one life of that age, after the longest of the two lives of B and C : and consequently, the value thereof being found, it is farther required to divide that reversion, properly between them.

Now the probabilities, that either one, or the other, of them will receive a payment in the first, second, third, &c. year, are (by quest. 101. vol. 2.) the continual products of the two probabilities of the dying of B and C , and the probability of the surviving of a person of the age of A , taken successively for those times.

Now if m and n represent the respective complements of the lives of B and C ; and p represents the probability of their being both extinct, at the end of one, two, or three years; then (by quest. 19.) $\frac{mp}{m+n}$ will exhibit the probability that the elder will within those times die before the younger; and $\frac{np}{n+m}$ the probability that the younger will so die before the elder; and these factors, $\frac{m}{m+n}$

and $\frac{n}{m+n}$, being to be applied to every probability, they will be constant factors, in every term of the series, which expresses the required probabilities.

If therefore the value of the reversion of an annuity, for the life of A , after the longest of the lives of B and C , be multiplied by $\frac{m}{m+n}$ the product will (when C is elder than B) be the value of A 's interest, in the annuity; and if the value of the same reversion, be multiplied by

$\frac{n}{m+n}$ the product will (upon the same supposition) be the value of D 's interest therein.

But, for as much as A and D are supposed to be of equal ages, the question will not be altered by supposing them to be but one person; and that the diversity consists, in that person's Interest, depending on the different survivorships of the elder, or the younger of the possessor's.

If A , B , and C , represent the values of the youngest, second, and eldest person, when the question will admit of the following variations, applicable to the three cases given in the last question.

CASE I.

If C , the eldest, is the expectant; B and A , the two younger, being possessors; then,

First, if C 's expectation depends upon A 's surviving B ;

then $\left(\frac{t}{t+m} \times D - \frac{n}{6r} \times \frac{nn}{4mt} \text{ or } \right)$

$D - \frac{n}{6r} \times \frac{nn}{t+m \times 4^t}$ will be the value of his interest in the reversion.

Secondly, if it depends upon B 's surviving A ;

then $\left(\frac{m}{t+m} \times D - \frac{n}{6r} \times \frac{nn}{4mt} \text{ or } \right)$

$D - \frac{n}{6r} \times \frac{nn}{t+m \times 4^t}$ will be that value.

COROL. I.

If the possessors are of equal ages; then $t=m$, and both

$\frac{t}{t+m}$ and $\frac{m}{t+m}$, will be equal to $\frac{1}{2}$; whence the value of C 's interest in the reversion will be

$\left(D - \frac{n}{6r} \times \frac{nn}{4mm} \times \frac{1}{2} \text{ or } \right) D - \frac{n}{6r} \times \frac{nn}{8mm}$.

EXAMPLE.

There is a copyhold estate held on the lives of two persons; *C* the father in possession, and *A* the son in expectation, of the ages 66 and 43; *A* has a wife, *B*, aged 54, who by the custom of the manor, will have her life in the premises, if her husband comes into possession; and she survives him: what is the present value of her interest in the estate?

By the example to case 2. quest. 29, the value of the reversion of an annuity for a life of 54, after the longest liver of two persons, of the ages 43 and 66, is 1,381;

also $\frac{t}{t+n} = \frac{43}{63}$; therefore $(1,381 \times \frac{43}{63}) = 0,942$

will be the value required.

Secondly, If *B*'s expectation depends upon *C*'s surviving *A*, then

$$\left(\frac{n}{t+n} \times \left\{ \frac{m}{6r} \times \frac{m}{2t} - \frac{n}{6r} \times 1 - \frac{n}{2m} \times \frac{n}{2t} \text{ or } \right. \right. \\ \left. \left. \frac{m}{6r} \times m - \frac{n}{6r} \right\} \times \frac{n}{t+n \times 2t} \right. \\ \left. \times 1 - \frac{n}{2m} \times n \right)$$

will be the value thereof.

EXAMPLE.

The same case being proposed, as in the last example, if (instead of *B*'s being the wife of the son, *A*) she be

84 MATHEMATICAL

the second wife of the father C; then, the value of her interest in the estate will be found, by multiplying (1,381) the value of the reversion of the annuity for a life of 54, after the longest liver of two persons, aged 43 and 66, by $\left(\frac{n}{t+n} = \right) \frac{20}{63}$; which product (viz.

$1,381 \times \frac{20}{63}$) is equal to 0.439;

And, if (0.942) the result in the former example be added to (0.439) the result in this; the sum (1,381) will be the value of the whole reversion.

COROL. I.

If the elder possessor C, and the expectant B, are of the same age; then $m = n$, and $\frac{m}{t} = \frac{n}{t}$, whence,

First, if the expectation depends upon A's surviving C;

then $\left(\frac{n}{6r} \times n - 1 - \frac{1}{2} \times n \times \frac{1}{t+n \times 2} \right)$ or

$\frac{n}{6r} \times \frac{n}{t+n \times 4}$ will be the value thereof.

Secondly, if it depends upon C's surviving A; then

$\left(\frac{n}{6r} \times n - 1 - \frac{1}{2} \times n \times \frac{n}{t+n \times 2t} \right)$ or

$\frac{n}{6r} \times \frac{nn}{t+n \times 4t}$ will be the value thereof.

COROL. II.

If the younger possessor A, and the expectant B, are of the same age; then $t = m$; whence

First,

First, if the expectation depends upon A's surviving C;

$$\text{then } \left\{ \frac{m}{6r} \times m - \frac{n}{6r} \right\} \times \frac{1}{m+n \times 2}$$

$$\times \left\{ 1 - \frac{n}{2m} \times n \right\}$$

will be the value thereof.

Secondly, if it depends upon C's surviving A; then

$$\left\{ \frac{m}{6r} \times m - \frac{n}{6r} \right\} \times \frac{n}{m+n \times 2m}$$

$$\times \left\{ 1 - \frac{n}{2m} \times n \right\}$$

will be the value thereof.

CASE III.

When the expectant A is younger than the possessors B, and C; then

First, if the expectation depends upon B's surviving C; then

$$\frac{m}{m+n} \times \left\{ \begin{aligned} & \frac{m}{6r} + \frac{n}{6r} \times \frac{m}{2t} \\ & - \frac{n}{6r} \times 1 - \frac{n}{2t} \times \frac{n}{2m} \end{aligned} \right\}$$

will be the value required.

Secondly, if it depends upon C's surviving B; then

$$\frac{n}{m+n} \times \left\{ \begin{aligned} & \frac{m}{6r} + \frac{n}{6r} \times \frac{m}{2t} \\ & - \frac{n}{6r} \times 1 - \frac{n}{2t} \times \frac{n}{2m} \end{aligned} \right\}$$

will be the value thereof.

COROL.

COROL.

If the possessors B and C are of equal ages; then $m = n$, and both $\frac{m}{m+n}$, and $\frac{n}{m+n}$, will be equal to $\frac{1}{2}$; whence the value of A 's interest in the reversion will be

$$\frac{1}{2} \times \frac{1}{r} + \frac{1}{2} \times \frac{1}{r} = \frac{1}{r} \times \frac{2}{2} = \frac{1}{r}.$$

QUESTION XXXI.

The respective ages of three persons, A , B , and C , being given; it is required to find the probability, that any two of them shall survive the third.

CASE I.

If one, or both, of the survivors are elder than the person to be survived; that is, if the complements of the survivors be n and t , that of the survived being m ; or if the complements of the survivors be n and m , that of the survived being t .

Then, the expectation of the joint lives of the survivors will be expressed, by the series $\frac{2n-1}{2n} \times \frac{2t-1}{2t} +$

$$\frac{2n-3}{2n} \times \frac{2t-3}{2t} (n); \text{ or, by the series } \frac{2n-1}{2n} \times \frac{2m-1}{2m} +$$

$$+ \frac{2n-3}{2n} \times \frac{2m-3}{2m} (n):$$

And (by arguing as in quest. 21.) the probabilities of their, severally, surviving the persons, whose complements are m and t , will be, $\frac{2n-1}{2n} \times \frac{2t-1}{2t} \times \frac{1}{m}$

$$+ \frac{2n-3}{2n}$$

+ $\frac{2n-3}{2n} \times \frac{2t-3}{2t} \times \frac{1}{n}$ (n) and $\frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{1}{n}$
 + $\frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{1}{n}$ (n); the sums of which
 will (by question 106, vol. 2.) be $\frac{n}{2m} = \frac{nn}{6mt}$ and $\frac{n}{2t}$
~~which were required.~~

CASE II.

If both the survivors be younger than the survivor.

Let the complements of the survivors be denoted by m and t ; and that of the survivor by n .

Then by arguing (as in quest. 22.) the probability required will be expressed by the series,

$$\frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{n} (n);$$

The sum of which, will, (by quest. 16.) be

$$\left(1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{3mt} \times \frac{1}{n} \text{ or } \right) 1 - \frac{n}{2m}$$

+ $\frac{n}{2t} - \frac{nn}{3mt}$; because $\frac{nn}{n}$ is a common factor, in every term of the series.

QUESTION XXXII.

A has (by his will) left a legacy of B pounds, if both he (B) and his wife (C) shall be alive at the time of his (A 's) decease; what is the present value of B 's expectation?

CASE I.

If either, or both, of the expectants be older than the survivor.

Then (by quest. 31. case, I.) the sum of the series of annual

annual probabilities will be, either $\frac{n}{2t} - \frac{nn}{6mt}$, or

$$\frac{n}{2m} - \frac{nn}{6mt};$$

That is, either $\frac{1}{t} \times \frac{n}{2} - \frac{nn}{6m}$, or $\frac{1}{m} \times \frac{n}{2} - \frac{nn}{6t}$;

consequently, the sum of the corresponding series of pre-

sent worths will be, either $\frac{1}{t} \times D - D - \frac{n}{6r} \times \frac{n}{2m}$,

$$\text{or } \frac{1}{m} \times D - D - \frac{n}{6r} \times \frac{n}{2t};$$

Whence, either $\frac{1}{t} \times D - D - \frac{n}{6r} \times \frac{n}{2m}$, or

$\frac{1}{m} \times D - D - \frac{n}{6r} \times \frac{n}{2t}$, will be the value required.

EXAMPLE I.

If *A* be aged 43 years; *B* and *C* being 66, and 54; and the legacy 25 £.

Then $t = 43$; $D = 25$;

And $D - D - \frac{n}{6r} \times \frac{n}{2m} = 6,276$, by example

3. quest. 4.

Therefore $\frac{6,276 \times 25}{43} = 3,649$ will be the value required.

EXAMPLE II.

If *A* be aged 54 years; *B* and *C* being 66 and 43; and the legacy 25 £.

Then $D - D - \frac{n}{6r} \times \frac{n}{2t} = 6,634$, by ex. 2. quest. 4.

And

And $\left(\frac{6,634 \times 25}{32} =\right)$ 5,183 will be the value required.

CASE II.

If both the expectants be younger than the possessor, then (by quest. 31. case 2.) the series of annual probabilities will be $\frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{n}$ (n.)

Now (by quest. 7.) the sum of n terms of the series of present worths, $\frac{2m-1}{2mr} \times \frac{2t-1}{2t} + \frac{2m-3}{2mr^2} \times \frac{2t-3}{2t}$

&c. is $\frac{2m-1}{2mr} \times \frac{2t-1}{2t} - \frac{nn}{6rm} \times \frac{n}{2t}$, which being multipl.

by $\frac{1}{n}$, the common factor of the above series of annual probabilities, will become $\frac{2m-1}{n} \times \frac{2t-1}{2t} - \frac{nn}{6rm} \times \frac{1}{2t}$;

Therefore $\frac{2m-1}{n} \times \frac{2t-1}{2t} - \frac{nn}{6rm} \times \frac{1}{2t}$ will be the value required.

EXAMPLE.

If A be aged 66; B and C , being 54 and 43; and the legacy 25£;

Then $\frac{2m-1}{n} \times \frac{2t-1}{2t} - \frac{nn}{6rm} \times \frac{1}{2t} = 8,056$, by quest.

7; and $\left(8,056 \times \frac{25}{20} =\right)$ 10,070 will be the value required.

SCHOLIUM.

Since the sum of the series of annual probabilities,

$$\frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{n} (n),$$

is $1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{3mt}$; and the sum of the corresponding series of present worths is

$$\frac{a}{n} - \frac{a}{2m} - \frac{a}{2t} + \frac{nn}{6mt} \times \frac{1}{2t}; \text{ Therefore, wherever}$$

the former appears in any operation, the value of the corresponding annuity, or survivorship, will be denoted by the latter.

QUESTION XXXIII.

The respective ages of *A*, *B*, *C*, and *D*, being given; it is required to find the probability, that any three of them shall survive the fourth.

CASE I.

If one or more of the survivors are elder than the person to be survived; that is (putting *t*, *m*, *n* and *v*, severally, for the complements of the youngest, second, third, and eldest) if the complements of the survivors are either *v*, *m*, and *t*; or *v*, *n* and *t*; or *v*, *n* and *m*; the complements of the person to be survived, being, severally, *n*, or *m*, or *t*.

Then (by arguing as in question 21 and 31.) the probabilities of the three survivorships will be expressed by the three series

$$\frac{2v-1}{2v} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{n} +$$

$$\left(\frac{2v-3}{2v} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{n} (v) \right),$$

2v-1

$$\begin{aligned} & \frac{2v-1}{2v} \times \frac{2n-1}{2n} \times \frac{2t-1}{2t} \times \frac{1}{m} + \\ & \left(\frac{2v-3}{2v} \times \frac{2n-3}{2n} \times \frac{2t-3}{2t} \times \frac{1}{m} (v) \right); \\ & \frac{2v-1}{2v} \times \frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{1}{t} + \\ & \left(\frac{2v-3}{2v} \times \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{1}{t} (v) \right); \end{aligned}$$

Which serieses have each a factor common to all their terms, and therefore their sums will (by quest, 107. vol.

$$\begin{aligned} 2.) \text{ be } & \frac{v}{2n} - \frac{vv}{6mn} - \frac{vv}{6tn} + \frac{v^3}{12mtn}, \\ & \frac{v}{2m} - \frac{vv}{6nm} - \frac{vv}{6tm} + \frac{v^3}{12mtm}, \end{aligned}$$

And $\frac{v}{2t} - \frac{vv}{6nt} - \frac{vv}{6mt} + \frac{v^3}{12nmt}$, which were required.

CASE II.

If all the survivors be younger than the person to be survived; then, putting n , m , and t , for the complements of the survivors, and v for that of the survived, the probability required will (by arguing as in question 33.) be expressed by the series,

$$\begin{aligned} & \frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{v} \\ & \left(+ \frac{2n-3}{2n} + \frac{2m-3}{2m} + \frac{2t-3}{2t} \times \frac{1}{v} (v) \right); \end{aligned}$$

The sum of which, will (if we multiply the result of quest. 17. by $\frac{1}{v}$, the constant factor in this series) appear to

$$\begin{aligned} \text{be } 1 - \frac{v}{2n} - \frac{v}{2m} - \frac{v}{2t} + \frac{vv}{3tn} + \frac{vv}{3tm} + \frac{vv}{3mn} - \\ \frac{v^3}{4nmt}; \text{ which is, therefore, the probability required.} \end{aligned}$$

QUES.

QUESTION XXXIV.

What is the present worth of a legacy of, or an estate worth, \mathcal{P} pounds, depending on the contingency of three persons surviving a fourth; the ages of those persons being given?

CASE I.

If one or more of the survivors are elder, than the person to be survived.

Then (by quest. 33. case 1. retaining the same symbols as are therein used) the sum of the series of annual probabilities will be, either $\frac{v}{2n} - \frac{vv}{6mn} - \frac{vv}{6tn} +$

$$\frac{v^3}{12mtn}, \text{ or } \frac{v}{2m} - \frac{vv}{6nm} - \frac{vv}{6tm} + \frac{v^3}{12ntm}, \text{ or } \frac{v}{2t} - \frac{vv}{6nt} - \frac{vv}{6mt} + \frac{v^3}{12nmt};$$

in the first of which $\frac{1}{n}$; in the second, $\frac{1}{m}$; and in the third, $\frac{1}{t}$, are constant factors:

therefore, if the present worths, corresponding to the probabilities, into which those common factors are multiplied, be severally multiplied by $\frac{\mathcal{P}}{n}$, $\frac{\mathcal{P}}{m}$, and $\frac{\mathcal{P}}{t}$; the results will (by quest. 9.) be

$$\mathcal{P} - \mathcal{P} - \frac{v}{6r} \times \frac{m+t \times 2 - v \times v}{4mt} \times \frac{\mathcal{P}}{n},$$

$$\mathcal{P} - \mathcal{P} - \frac{v}{6r} \times \frac{n+t \times 2 - v \times v}{4nt} \times \frac{\mathcal{P}}{m},$$

$$\text{and } \mathcal{P} - \mathcal{P} - \frac{v}{6r} \times \frac{n+m \times 2 - v \times v}{4nm} \times \frac{\mathcal{P}}{t}$$

the values required.

CASE

CASE II.

If all the survivors be younger, than the person to be survived; then (by case 2. quest. 33.) the series of annual probabilities will be

$$\frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{v} + \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{v} (v); \text{ and (by quest. 10.) the sum of } v \text{ terms of the series of present worths, } \frac{2n-1}{2nr} \times \frac{2m-1}{2mr} \times \frac{2t-1}{2tr} + \frac{2n-3}{2nr^2} \times \frac{2m-3}{2mr} \times \frac{2t-3}{2tr} \&c. \text{ is } \frac{1}{v} -$$

$$\left. \begin{aligned} & \frac{v}{6rn} \times m + t \times 2 - v \\ & - \frac{vv}{6rn} \times n - v \times 2 \end{aligned} \right\} \times \frac{v}{4mt};$$

if, therefore, this expression be multiplied by $\frac{P}{v}$ the result, viz.

$$\left. \begin{aligned} & \frac{P}{v} - \frac{vv}{6rn} \times m + t \times 2 - v \\ & - \frac{vv}{6rn} \times n - v \times 2 \end{aligned} \right\} \frac{v}{4mt} \times \frac{P}{v}$$

will be the value required.

EXAMPLE.

D, aged 74, hath, by his will, left a legacy of 25*£*. to *C*, his brother, aged 66; provided he, *C*; his wife, *B*, aged 54; and his daughter, *A*, aged 43; shall be, all of them, alive at the time of his, *D*'s, decease: what is the value of *C*'s interest in the legacy? By

94. MATHEMATICAL

By the example to quest. 10. the expression

$$\left. \begin{aligned} & \frac{vv}{6rn} \times m + t \times 2 - v \\ & - \frac{vv}{6rn} \times n - v \times 2 \end{aligned} \right\} \times \frac{v}{4mt} = 4,870;$$

And $(4,870 \times \frac{25}{10} =) 12,175$ will be the value of C's interest in the legacy.

SCHOLIUM.

Since the sum of the series of annual probabilities,

$$\begin{aligned} & \frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{v} + \frac{2n-3}{2n} \times \frac{2m-3}{2m} \\ & \times \frac{2t-3}{2t} \times \frac{1}{v} (v), \text{ is } 1 - \frac{v}{2t} - \frac{v}{2m} - \frac{v}{2n} + \\ & \left[\frac{vv}{3tm} + \frac{vv}{3tn} + \frac{vv}{3mn} - \frac{v^3}{4tmn} \right], \text{ and since the sum of} \\ & \text{the corresponding series of present worths is } \frac{vv}{v} \end{aligned}$$

$$\left. \begin{aligned} & \frac{vv}{6rn} \times m + t \times 2 - v \\ & - \frac{vv}{6rn} \times n - v \times 2 \end{aligned} \right\} \times \frac{1}{4mt};$$

therefore, whenever the former occurs in any operation, the value of the corresponding annuity, or survivorship, will be denoted by the latter.

QUESTION XXXV.

The respective ages of three persons *A*, *B*, and *C* (of whom *A* is the youngest, and *C* the eldest) being given; it is required to find the probability, that *A* shall die first, *B* next, and *C* last.

SOLU.

SOLUTION.

Let the complements of A , B , and C , be severally denoted by t , m , and n : then, since $\frac{2t-1}{2t}$ is the expectation

of A 's life for the first year, $1 - \frac{2t-1}{2t}$ will be the expectation of his death, at the beginning, or early in that year. If, therefore, this expectation of A 's death be multiplied by, $\frac{2n-1}{2n} \times \frac{1}{m}$, the probability of C 's surviving B , for that year (see quest. 21.) then the product,

$1 - \frac{2t-1}{2t} \times \frac{2n-1}{2n} \times \frac{1}{m}$, will exhibit the probability of A 's dying first, and B next, in the first year, and that C shall survive them both for that year: and (by arguing in the same manner) the probabilities of the like event's taking place, in the second, third, &c.

year, will be $1 - \frac{2t-3}{2t} \times \frac{2n-3}{2n} \times \frac{1}{m}$,

$1 - \frac{2t-5}{2t} \times \frac{2n-5}{2n} \times \frac{1}{m}$, &c. and consequent-

ly the probability required will be the sum of this series of products.

Now by actual multiplication these products will become

$$\frac{2n-1}{2n} \times \frac{1}{m} - \frac{2t-1}{2t} \times \frac{2n-1}{2n} \times \frac{1}{m},$$

$$\frac{2n-3}{2n} \times \frac{1}{m} - \frac{2t-3}{2t} \times \frac{2n-3}{2n} \times \frac{1}{m},$$

$$\frac{2n-5}{2n} \times \frac{1}{m} - \frac{2t-5}{2t} \times \frac{2n-5}{2n} \times \frac{1}{m};$$

&c.

&c.

Where

Where the affirmative series, $\frac{2n-1}{2n} \times \frac{1}{m} + \frac{2n-3}{2n} \times \frac{1}{m}$
 &c. is the probability of *C*'s surviving *B* (by quest. 21.)
 and the negative series, $\frac{2t-1}{2t} \times \frac{2n-1}{2n} \times \frac{1}{m} +$
 $\frac{2t-3}{2t} \times \frac{2n-3}{2n} \times \frac{1}{m}$, &c. is the probability that both
A and *C* will survive *B*, by quest. 31, case 1.

Hence, if we call the person proposed to die first, the
possessor; the person who is to die next, the *expectant*;
 and the third person the *survivor*.

Then, from the probability of the survivor's out-living the
expectant; take the probability, that both survivor and
possessor shall out-live the expectant; and the remainder will
be the probability of the proposed order of survivorship,
among the three persons.

Now the probability of *C*'s surviving *B* is $\frac{n}{2m}$ (by qu.
 21.) and the probability, that both *A* and *C* shall survive
B, is $\frac{n}{2m} - \frac{nn}{6tm}$ (by case 1. quest. 31.) therefore, their
 difference, $\frac{nn}{6tm}$, will be the probability required.

COROL. I.

If the ages of the possessor *A*, and expectant *B*, are
 equal; then $t = m$; and the probability required will
 become $\frac{nn}{6mm}$.

COROL. II.

If the ages of the expectant *B*, and survivor *C*, are
 equal; then $m = n$; and $\left(\frac{nn}{6tn} \text{ or } \right) \frac{n}{6t}$ will be the pro-
 bability required.

COROL.

COROL. III.

If the three persons are of equal ages; then $t=m=n$; and the probability will become $\left(\frac{nn}{6nn} = \right) \frac{1}{6}$.

QUESTION XXXVI.

The same lives being considered, as in the last question; it is required to find the probability, that B , shall die first; A , the youngest, next; and C , the eldest, last?

SOLUTION.

Here the complement of the possessor, is m ; of the expectant, t ; and of the survivor, n .

From $\frac{n}{2t}$ the probability of C 's surviving A ,

Take $\frac{n}{2t} - \frac{nn}{6mt}$; the probability that B and C shall survive A ;

Remains, $\frac{nn}{6mt}$, for the probability of the given order of survivorship; which probability is the same, with the result of the last question.

COROL. I.

If the ages of the possessor, B , and expectant A , are equal; then $t=m$; and the answer will be $\frac{nn}{6mm}$, as in corol. 1. quest. 35.

COROL. II.

If the ages of the possessor, B , and survivor, C , are equal; then $m=n$; and the result will be $\left(\frac{nn}{6nt} =\right)$

$\frac{n}{6t}$, as in corol. 2. quest. 35.

QUESTION XXXVII.

In the same lives as before; it is required to determine the probability, that A , the youngest, shall die first; C , the eldest, next; and B , last.

Here the complement of the possessor, is t ; of the expectant, n ; and of the survivor, m .

The probability of the survivor's outliving the expectant will (by quest. 22.) be $1 - \frac{n}{2m}$;

And the probability, that both possessor and survivor shall outlive the expectant, is (by case 2. question 31.)

$$1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{3mt};$$

Which, being subtracted from the former, leaves $\frac{n}{2t} - \frac{nn}{3mt}$, for the probability required.

COROL. I.

If the possessor, A , and the survivor, B , be of equal ages; then $t=m$; and, $\frac{n}{2m} - \frac{nn}{3mm}$, will be the probability required.

COROL.

COROLL. II.

If the expectant, C , and survivor, B , be of equal ages; then $m = n$; and the answer will become $\frac{n}{2t} - \frac{nn}{3nt}$;

That is $\left(\frac{n}{2t} - \frac{n}{3t}\right) = \frac{n}{6t}$, as in corol. 2. quest. 35 and 36.

QUESTION XXXVIII.

In the same lives as before; required the probability, that C , the eldest, shall die first; A , the youngest, next; and B , last.

Here, the complement of the possessor, is n ; of the expectant, t ; and of the survivor, m .

Therefore from $\frac{n}{2t}$;

Take $\frac{n}{2t} - \frac{nn}{6mt}$;

And the remainder $\left(\frac{n}{2t} - \frac{n}{2t} + \frac{nn}{6mt}\right)$ or

$m - n + \frac{nn}{3m} \times \frac{1}{2t}$ will be the probability required.

COROLL. I.

If the possessor C , and survivor B , are of equal ages; then $m = n$; and $\frac{n}{2t} - \frac{n}{2t} + \frac{nn}{6nt}$ or $\frac{n}{6t}$ will be the answer, as in corol. 2. quest. 35, 36, and 37.

COROL. II,

If the expectant, A , and survivor, B , are of equal ages; then $t = m$; and $\left(\frac{m}{2m} - \frac{n}{2m} + \frac{nn}{6mm} \text{ or } \right)$

$\frac{1}{2} - \frac{n}{2m} + \frac{nn}{6mm}$ will be the answer.

QUESTION XXXIX.

In the same lives as before; it is required to determine the probability, that B , shall die first; C the eldest, next; and A the youngest, last.

Here the complement of the possessor is m ; of the expectant, n ; and of the survivor, t .

The probability of the survivor's outliving the expectant is (by quest. 22) $1 - \frac{n}{2t}$; and the probability, that both possessor and survivor will outlive the expectant, is (by case 2. quest. 31) $1 - \frac{n}{2m} - \frac{n}{2t} + \frac{nn}{3mt}$; which

being subtracted from the former, leaves, $\frac{n}{2m} - \frac{nn}{3mt}$ for the probability required.

COROL. I.

If the possessor B , and the expectant C , are of equal ages; then $m = n$; and $\left(\frac{n}{2n} - \frac{nn}{3nt} \text{ or } \right) \frac{1}{2} - \frac{n}{3t}$ will be the answer.

COROL.

COROLL., II.

If the possessor *B*, and survivor *A*, are of equal ages; then $t = m$; and $\frac{n}{2m} - \frac{nn}{3mm}$ will be the probability required, as in corol. I. quest. 37.

QUESTION XL.

In the same lives as before; it is required to determine the probability, that *C*, the eldest, shall die first; *B*, next; and *A*, the youngest, last.

Here the complement of the possessor is n ; of the expectant, m ; and of the survivor, t .

Then from $1 - \frac{nn}{2t}$;

Take $\frac{n}{2m} - \frac{nn}{6tm}$;

The remainder, $1 - \frac{n}{2t} - \frac{n}{2m} + \frac{nn}{6tm}$,

will be the probability required.

COROL. I.

If the ages of the possessor *C*, and expectant *B*, are equal; then $m = n$; and the answer will be $1 - \frac{n}{2t} - \frac{n}{2n} + \frac{nn}{6nt}$;

That is $(1 - \frac{1}{2} - \frac{n}{2t} - \frac{n}{6t}) = \frac{1}{2} - \frac{n}{3t}$; as in corol. I. quest. 39.

COROL. II.

If the ages of the expectant, B , and survivor, A , be equal; then $t = m$; and the probability required will become $1 - \frac{m}{2m} - \frac{n}{2m} + \frac{nn}{6mm}$;

That is $(1 - \frac{1}{2} - \frac{n}{2m} + \frac{nn}{6mm}) - \frac{1}{2} - \frac{n}{2m} + \frac{nn}{6mm}$; as in corol. 2. quest. 38.

The results of the last 6 questions and their corollaries are inserted in the following table; together with the numerical answers, when the complements, t , m , and n , denote severally, 43, 32 and 20.

A TABLE

A TABLE of the probabilities of all the several orders of survivorship, that can possibly happen, among three persons; *C*, the eldest; *B*, the second; and *A*, the youngest.

SOLUTION.				Put x , for the complement of the eldest life; m , for that of the second; and t , for that of the youngest.	Numerical ant. for the ages 66, 54, and 43.
Case	Proportion	Expectant	If the ages are		
1	A	B	C all unequal	$\frac{mt}{Otm}$	0,0484
2	B	A	C all unequal	$\frac{mt}{Otm}$	0,0484
3	A	C	B all unequal	$\frac{n}{2t} - \frac{mt}{3tm}$	0,1358
4	C	A	B all unequal	$\frac{m}{2t} - \frac{n}{2t} + \frac{mt}{6tm}$	0,1879
5	B	C	A all unequal	$\frac{n}{2m} - \frac{mt}{3tm}$	0,2157
6	C	B	A all unequal	$1 - \frac{m}{2t} - \frac{n}{2m} + \frac{mt}{6tm}$	0,3638

Case	Portion		Expectant	Survivor	If the ages are	The sum of the above 6 probabilities, taken either lit- terally, or figurally, is unity; and they are the same, with those given by Mr. De Moivre, in his treatise of annuities	Numerical answer for ages 66, 54, and 42.
	A	B					
7	A	B	C	A	B	$\frac{m}{6m}$	0,0651
8	A	B	C	A	B	$\frac{n}{6l}$	0,0775
9	A	C	B	A	B	$\frac{n}{2m}$	0,1823
10	A	C	B	A	B	$\frac{n}{2m}$	0,2526
11	A	C	B	A	B	$\frac{1}{2}$	0,345
12	A	C	B	A	B	$\frac{1}{6}$	0,1667

QUEST.

CASE III.

When the survivor is younger, than the two persons to be survived.

This survivorship will take place, when the three persons die in the order B, C, A ; or in the order C, B, A .

Let, therefore, the complement of the survivor's life be denoted by t ; and those of the two persons, to be survived, by n , and m .

Then (by quest. 39.) the probability of their dying in the order B, C, A , will be $\left\{ \frac{n}{2m} - \frac{nm}{3tm} \right\}$

And (by quest. 40.) the probability of their dying in the order C, B, A , will be $\left\{ 1 - \frac{m}{2t} - \frac{n}{2m} + \frac{nm}{6tm} \right\}$

Therefore, their sum, $1 - \frac{m}{2t} - \frac{nm}{6tm}$, will be the probability required.

COROL. I.

If the two persons, that are to be survived, are of equal ages; that is, if $B = C$; then the probability required will be $\left(1 - \frac{n}{2t} - \frac{nm}{6tn} = 1 - \frac{n}{2t} - \frac{n}{6t} \right)$

$$1 - \frac{2n}{3t}$$

COROL. II.

If the survivor, and the younger of the persons to be survived, are of equal ages; that is, if $A = B$; then the answer will become $\left(1 - \frac{m}{2m} - \frac{nm}{6mm} = 1 - \frac{1}{2} - \frac{n}{6m} \right)$

$$= \frac{1}{2} - \frac{n}{6mm} \text{ as in corol. 1. case 2.}$$

These survivorships are inserted in the following table, in the same manner as the former.

108 MATHEMATICAL

A TABLE of the probabilities of the survivorship, of any one, out of three persons; viz, *A*, the youngest; *B*, the second; and *C*, the eldest.

Case	Persons to be survived.	If the lives are	SOLUTION: Put x , for the complement of the eldest life; m , for that of the second; and t , for that of the youngest.	Numerical values for the ages 43, 54, and 66.
1	<i>A, B, C</i> all unequal	all unequal	$\frac{m}{x} - \frac{m}{x}$	0, 0968
			$\frac{m}{x} - \frac{m}{x}$	
2	<i>A, C, B</i> all unequal	all unequal	$1 - \frac{m}{2t}$	0, 3237
			$1 - \frac{m}{2t}$	
3	<i>B, C, A</i> all unequal	all unequal	$1 - \frac{m}{2t}$	0, 5795
			$1 - \frac{m}{2t}$	

Case	Persons to be survived.	Survivor	If the lives are	The sum of the above three probabilities, taken either literally or numerally, is unity; and they are the same with those given by Mr. De Moivre, in his treatise of annuities.	Numerical anl. for the ages 43, 54, and 66.
4	A, B, C	C	$A = B$	$\frac{n}{2}$	0, 1302
5	A, B, C	C	$B = C$	$\frac{n}{3}$	1550
6	A, B, C	C	$A = B$	$\frac{1}{2} - \frac{n}{20000}$	0, 4349
7	B, C, A	A	$B = C$	$1 - \frac{2}{3}$	0, 6900
8	A, B, C A, C, B B, C, A	C B A	all equal	$\frac{1}{3}$	0, 3333

QUESTION XLII.

The respective ages of three persons, A , B , and C (whereof A is the youngest, and C the eldest) are given;

A is possessed of an estate, worth P pounds, which upon his decease will devolve to B , if then living, for his life only; after which C (if he survives B) is to possess it, without farther limitation; but, if B dies before A , then C is not to inherit at all; what is the present worth of C 's interest in that estate?

SOLUTION.

If t , m , and n , severally, represent the complements of life of A , B , and C ; and P denotes the value of an annuity (secured by land) for the life of C .

Then, because $\left(\frac{nn}{6tm}\right) = \frac{1}{m} \times \frac{nn}{6t}$ is (by quest. 35.) the sum of the series of the annual probabilities, that the proper order of survivorship will take place; therefore $\left(\frac{1}{m} \times P - \frac{n}{6r} \times \frac{n}{2t} \text{ or } \right) P - \frac{n}{6r} \times \frac{n}{2tm}$ will be the sum of the series of present worths, corresponding thereto.

And, since P is the value of the estate in question, therefore $P - \frac{n}{6r} \times \frac{n}{2tm}$ will be the value of C 's interest therein; which was required.

EXAMPLE.

C (aged 66) will enjoy an estate of 1*l.* per annum, or a legacy of 25*l.* if A (aged 43) dies before B (aged 54); and B also dies before him (C); required the present value of his interest therein?

Here

Here $P - \frac{n}{6r} = 4,468$; $n=20$; $m=32$; $t=43$; and $P = 25$;

Therefore $\frac{4,468 \times 20 \times 25}{2 \times 43 \times 32} = 0,812$ will be the value required.

COROL. I.

If the possessor, and expectant, are of equal ages; then $t = m$; and the interest of the survivor will be-

$$\text{come } P - \frac{n}{6r} \times \frac{nP}{2mn} = 1,091.$$

COROL. II.

If the expectant, and survivor, are of equal ages; then $m = n$; and the interest of the survivor will become,

$$(P - \frac{n}{6r} \times \frac{nP}{2tn}) = P - \frac{n}{6r} \times \frac{P}{2t} = 1,299.$$

COROL. III.

If the three persons are of equal ages; then $t=m=n$; and the interest of the survivor will become,

$$(P - \frac{n}{6r} \times \frac{nP}{2nn}) = P - \frac{n}{6r} \times \frac{P}{2n} = 2,7925.$$

QUESTION XLII.

The same things being given, as in the last question; suppose B , to be the possessor; and A (the youngest) to be the expectant; and, that the interest of C (the eldest) depending on B 's dying before A , and A 's dying before C , is required.

SOLU.

SOLUTION.

If the same symbols be retained as in the last question; then $\left(\frac{nn}{6mt} \text{ or } \right) \frac{1}{t} \times \frac{nn}{6m}$ being (by quest. 16.) the sum of the series of annual probabilities; therefore

$\left(\frac{1}{t} \times \mathcal{P} - \frac{n}{6r} \times \frac{n}{2mt} \text{ or } \right) \mathcal{P} - \frac{n}{6r} \times \frac{n}{2mt}$ will be the sum of the corresponding series of present worths; and consequently, $\mathcal{P} - \frac{n}{6r} \times \frac{n}{2mt}$ will be the value of C's interest; the same as in the last question.

COROL. I.

Hence, if the possessor and expectant are of equal ages; the answer will be $\mathcal{P} - \frac{n}{6r} \times \frac{n}{2mt}$; as in corol. 1. of the last question.

COROL. II.

If the possessor and survivor are of equal ages; the answer will be $\mathcal{P} - \frac{n}{6r} \times \frac{n}{2t}$ as in corol. 2. of the last quest.

QUESTION XLIV.

The same things being given, as in the two preceding questions; suppose A (the youngest) to be the possessor; C (the eldest) to be the expectant; and that the interest of

B,

B, in the estate (depending upon *A*'s dying before *C*, and *C*'s dying before *B*) be required.

SOLUTION.

The probability of this order of survivorship, being (in quest. 37.) found by subtracting $1 - \frac{n}{2m} - \frac{n}{2t}$

+ $\frac{nn}{3mt}$ (the sum of n terms of the series of annual probabilities $\frac{2m-1}{2m} \times \frac{2t-1}{2t} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{2t-3}{2t} \times \frac{1}{n}$

&c.) from $1 - \frac{n}{2m}$ (the sum of n terms of the series of

annual probabilities $\frac{2m-1}{2m} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{1}{n}$ &c.)

Th. the value of the survivorship must (by the scholiums annexed to quest. 23. (case 2) and 32) be found by sub-

tracting $\frac{\frac{nn}{3mt}}{n} - \frac{\frac{nn}{6rm}}{n} \times \frac{1}{2t}$ (the sum of n terms

of the series of present worths, corresponding to the

former) from $\frac{\frac{nn}{3mt}}{n}$ (the sum of n terms of the series of

present worths, corresponding to the latter;) and there-

fore, if $\frac{\frac{nn}{6rm}}{n} \times \frac{1}{2t}$ (their difference) be mul-

tiplied by \mathcal{P} (the value of the estate) the product,

$\frac{\frac{nn}{6rm}}{n} \times \frac{\mathcal{P}}{2t}$, will be the value of *B*'s interest

therein.

EXAMPLE.

B (aged 54) the second wife of *C* (aged 66) will enjoy an estate of 1£. per annum, or a legacy of 25£. if *A* (aged

114 MATHEMATICAL

(aged 43) the daughter of C, should die before her father, and she (B) should survive him; required the present value of B's interest therein.

Here $\frac{P}{6rm} - \frac{nn}{6rm} = 7,888$ (by exa. to quest. 7.) $P = 25$; and $t = 43$.

Then $\frac{7,888 \times 25}{86} = 2,293$ will be the value of B's interest.

COROL. I.

If the possessor and survivor are of equal ages; then $t = m$; and the value of B's interest will become

$$\frac{P}{6rm} - \frac{nn}{6rm} \times \frac{P}{2m} = 3,081.$$

COROL. II.

If the expectant and the survivor are of equal ages; then $m = n$; and $\frac{P}{6rm} - \frac{nn}{6rm}$; whence the value of B's interest will become $(\frac{P}{6rm} - \frac{nn}{6rm} \times \frac{P}{2n})$ or

$$\frac{P}{6r} - \frac{n}{6r} \times \frac{P}{2n} \text{ as in corol. 2. quest. 42.}$$

QUESTION XLV.

The same things being given, as in the preceding questions; suppose C (the eldest) to be the possessor; A (the youngest) to be the expectant; and that the interest of, B, in the estate (depending upon C's dying before A, and A's dying before B) be required.

S O L U.

SOLUTION.

Retaining the usual symbols, the sum of the series of the annual probabilities of this order of survivorship is

(by quest. 38. $\frac{m}{2t} - \frac{n}{2t} + \frac{mn}{6mt}$ or)

$\frac{1}{t} \times \frac{m}{2} - \frac{n}{2} + \frac{mn}{6m}$; therefore the sum of the corresponding series of present worths will be

$\frac{1}{t} \times 99 - 99 + 99 - \frac{n}{6r} \times \frac{n}{2m}$; and consequently,

ly, the present value of B's interest in the estate will be

$\frac{1}{t} \times 99 - 99 + 99 - \frac{n}{6r} \times \frac{n}{2m}$

EXAMPLE.

C (aged 66) who is possessed of an estate of 1 $\frac{1}{2}$. per annum, has by deed given it to his wife A (aged 43) if she be alive at the time of his decease; required the interest of B, aged 54 (who is brother, and heir at law, to the wife) in the estate, dependent on her surviving the husband, and the brother's surviving her.

Here $99 = 10,757$; $99 = 7,673$; $99 - \frac{n}{6r} \times \frac{n}{2m} = 1,397$ (by exa. 3. quest. 4.) $99 = 25$; and $t = 43$.
Then $10,757 - 7,673 + 1,397 = 4,481$ and
 $\left(\frac{4,481 \times 25}{43} \right) = 2,605$ will be the value required.

COROL. I.

If the expectant and survivor are of equal ages; then $t = m$; and the interest of B will become

99

$$\frac{P}{m} \times Q - P + P - \frac{n}{6r} \times \frac{n}{2m} = 3,501.$$

COROL. II.

If the possessor and survivor are of equal ages; then $m = n$, and $Q = P$; therefore the interest of B will

$$\text{become } \left(\frac{P}{t} \times P - P + P - \frac{n}{6r} \times \frac{n}{2n} = \right.$$

$$\frac{P}{t} \times P - \frac{n}{6r} \times \frac{1}{2} = P - \frac{n}{6r} \times \frac{P}{2t} \text{ as in corol.}$$

2. quest. 42.

QUESTION XLVI.

The same things being given, as in the preceding questions; suppose B to be the possessor; C (the eldest) to be the expectant; and that the interest of A (the youngest) in the estate (depending upon B 's dying before C , and C 's dying before A) be required;

SOLUTION.

The probability of this order of survivorship being (in quest. 39.) found by subtracting $1 - \frac{n}{2m} - \frac{n}{2t} +$

$\frac{nm}{3mt}$, from $1 - \frac{n}{2t}$; therefore the value required must

be found, by subtracting $\frac{n}{2} - n \cdot m - \frac{nm}{6rm} \times \frac{P}{2t}$

from $\frac{nP}{2}$;

And

And therefore, if the remainder, $\frac{P}{n} - \frac{M}{n} +$

$\frac{M}{n} - \frac{nn}{6rm} \times \frac{1}{2t}$; be multiplied by, P , the value of the estate; the product, viz.

$P \times \frac{P}{n} - \frac{M}{n} + \frac{M}{n} - \frac{nn}{6rm} \times \frac{1}{2t}$, will be the present value of A 's interest therein.

EXAMPLE.

B (aged 54) is possessed of an estate of 1000 £ . per annum; which he has, by deed, settled on his sister C (aged 66) if she survive him; but if she dies before him, her son A (aged 43) is not to inherit; required the present value of A 's expectation, dependent on C 's surviving B , and A 's surviving C .

Here $P = 10,838$; and $M = 9,891$ (by quest. 2.)

$\frac{M}{n} - \frac{nn}{6rm} = 7,888$ (by quest. 7.) $n = 20$; $t = 43$;

and $P = 25$. Then $\frac{10,838 - 9,891}{20} = 0,04735$; and

$\frac{7,888}{2 \times 43} = 0,09172$;

Therefore $(0,04735 + 0,09172 \times 25 =) 3,477$ will be the value required.

COROL. I.

If the possessor and survivor are of equal ages; then $P = M$, and $t = n$; whence the value of A 's interest will become

$$\left(P \times \frac{P}{n} - \frac{M}{n} + \frac{M}{n} - \frac{nn}{6rm} \times \frac{1}{2m} \right) =$$

$\frac{P}{n} - \frac{nn}{6rm} \times \frac{1}{2m}$ as in corol 1. quest. 44.

COROL.

COROL. II.

If the possessor and expectant are of equal ages; then $m = n$, and $m = n$; whence the value of A 's interest will become
$$P \times \frac{a^x - P}{n} + P - \frac{nr}{6r} \times \frac{1}{2t}$$

$$=) P \times \frac{a^x - P}{n} + P - \frac{n}{6r} \times \frac{1}{2t} = 5,255.$$

QUESTION XLVII.

The same things being given, as in the preceding questions; suppose C (the eldest) to be the possessor; B , to be the expectant; and that the interest of A (the youngest) in the estate (depending upon C 's dying before B , and B 's dying before A) be required.

SOLUTION.

The same symbols being retained as before; since $1 - \frac{m}{2t} - \frac{n}{2m} + \frac{mn}{6tm}$ is (by quest. 40.) the sum of the series of the annual probabilities of this order of survivorship; and since $\frac{a^x}{m}$, $\frac{P}{m}$, and $P - \frac{n}{6r} \times \frac{n}{2tm}$ are the sums of the three series of present worths, severally, corresponding to the three series of annual probabilities, whose sums are, $1 - \frac{m}{2t}$, $\frac{n}{2m}$, and $\frac{mn}{6tm}$;

$$\text{Therefore } P \times \frac{a^x}{m} - \frac{P}{m} + P - \frac{n}{6r} \times \frac{n}{2tm}$$

or)

or) $\frac{P}{m} \times m\bar{x} - P + P - \frac{n}{6r} \times \frac{n}{2t}$ will be the value of A 's interest in the estate.

EXAMPLE.

C (aged 66) is possessed of an estate of 1*l.* per annum, which will descend to his brother B (aged 54) if he be alive at the time of C 's decease; it is required to determine, what interest A (the wife of B , aged 43) has in the estate, dependent on B 's surviving C , and A 's surviving B ?

Here $m\bar{x} = 12,578$; $P = 7,673$; $P - \frac{n}{6r} \times \frac{n}{2t} = 1,039$, $P = 25$; and $m = 32$:

Therefore $(12,578 - 7,673 + 1,039 \times \frac{32}{32}) = 4,644$ will be the value required.

COROL. I.

If the expectant and survivor are of equal ages; then $m\bar{x} = P$, and $t = m$; therefore the interest of A will

become $\frac{P}{m} \times P - P + P - \frac{n}{6r} \times \frac{n}{2m}$ as in corol. 1. quest. 45.

COROL. II.

If the possessor and expectant are of equal ages; then $m = n$; and the value of A 's interest in the estate will be

$$\left(\frac{P}{n} \times m\bar{x} - P + P - \frac{n}{6r} \times \frac{n}{2t} = \right)$$

$P \times \frac{m\bar{x} - P}{n} + P - \frac{n}{6r} \times \frac{1}{2t}$ as in corol. 2. quest. 46.

QUES.

QUESTION XLVIII.

To find the present worth of the reversion of an estate, worth \mathcal{P} pounds, depending on one person's surviving two others.

SOLUTION.

Let A , B , and C , severally, represent the name of the younger, second, and elder person; t , m , and n , their complements of life; \mathcal{P} and \mathcal{Q} the values of annuities (secured by land) for the lives of B and C ; ${}^m\mathcal{P}$, ${}^n\mathcal{P}$, and ${}^n\mathcal{Q}$, the values of such annuities for m , and n , years certain, if A and B should live so long.

CASE I.

If the survivor be elder than the two persons to be survived; that is, if C is to survive both A and B .

Then, if A be supposed to die first; and B , second; the value of C 's interest in the estate will (by quest. 42.) be

$$\mathcal{P} - \frac{n}{6r} \times \frac{{}^n\mathcal{P}}{2tm}; \text{ and if } B \text{ be supposed to die first;}$$

and A second; the value of C 's interest will also be

$$\mathcal{Q} - \frac{n}{6r} \times \frac{{}^n\mathcal{Q}}{2tm}; \text{ by quest. 43.}$$

Therefore their sum, $\mathcal{P} - \frac{n}{6r} \times \frac{{}^n\mathcal{P}}{tm}$, will be the value required.

EXAMPLE.

C (aged 66) will become possessed of an estate of 1*l.* per annum, if he survives his two neices A , and B , of the respective ages of 43, and 54; required the present value of his interest in that estate.

Here

Here $\mathcal{A} - \frac{n}{6r} \times \frac{n\mathcal{P}}{2tm} = 0,812$ by example to quest. 42.

Therefore $(0,812 \times 2 =) 1,624$ will be the value required.

COROL. I.

If the persons, to be survived, are of equal ages.

Then $t = m$; and the value will become

$$\mathcal{A} - \frac{n}{6r} \times \frac{n\mathcal{P}}{mm} = 2,182.$$

COROL. II.

If one of the persons, to be survived, be of the same age with the survivor; then $m = n$; and the value will

become $\left(\mathcal{A} - \frac{n}{6r} \times \frac{n\mathcal{P}}{tn} = \right) \mathcal{A} - \frac{n}{6r} \times \frac{\mathcal{P}}{t} = 2,598.$

COROL. III.

If all the three persons are of equal ages; then $t = m = n$, and the value required will be $\mathcal{A} - \frac{n}{6r} \times \frac{\mathcal{P}}{n} = 5,585.$

CASE II.

If the survivor be, younger than the one, and elder than the other, of the two persons to be survived; that is, if B is to survive C and A .

Then if A , be supposed to die first; and C , second; the value of B 's interest will (by quest. 44.) be

$\frac{P}{6rm} \times \frac{P}{2t}$: and if, G be supposed to die first; and A , second; B 's interest will by quest. 45. be worth $(P - P + P - \frac{P}{6r} \times \frac{P}{2m} \times \frac{P}{t} =)$

$$P - P \times 2 + P - \frac{P}{6r} \times \frac{P}{m} \times \frac{P}{2t}$$

Therefore, their sum

$$(P - P \times 2 + P - \frac{P}{6r} \times \frac{P}{m} + P - \frac{P}{6rm} \times \frac{P}{2t}$$

$$\text{or}) P - P \times 2 + P + P - \frac{2P}{6r} \times \frac{P}{m} \times \frac{P}{2t}$$

will be the value required.

EXAMPLE.

C (aged 66) is possessed of an estate of 1 £. per annum, which will descend to his daughter, A (aged 43); but after the decease of both of them, will become the property of B , the brother of C , aged 54, if he be then living; required the value of B 's interest in that estate.

Here $P = 10,757$; $P = 7,673$; $P = 9,891$; $\frac{P}{6r} = 3,205$; $n = 20$; $m = 32$; $P = 25$; and $t = 43$;

Then $10,757 - 7,673 \times 2 = 6,168$; $3,205 \times 2 = 6,410$; $7,673 - 6,410 = 1,263$; $P - \frac{2P}{6r} \times \frac{P}{m} = 1,263 \times 20 = 0,789$; and $6,168 + 9,891 + 0,789 = 16,848$.

Therefore $(\frac{16,848 \times 25}{2 \times 43} =)$ 4,898 will be the value required.

COROLL.

COROL. I.

If the survivor and the younger person, to be survived, are of the same age; then $t = m$; and the value required will be,

$$P - P \times 2 + P + P - \frac{2n}{6r} \times \frac{n}{m} \times \frac{P}{2m} \\ = 6,581.$$

COROL. II.

If the survivor and the elder person, to be survived, are of the same ages; then $m = n$, and $m = n$; whence the value required will be

$$P - P \times 2 + P + P - \frac{2n}{6r} \times \frac{n}{n} \times \frac{P}{2t};$$

$$\text{That is } (2P - \frac{2n}{6r} \times \frac{P}{2t} \text{ or }) P - \frac{n}{6r} \times \frac{P}{t}$$

as in corol. 2. case 1.

CASE III.

If the survivor be younger, than either of the persons, to be survived; that is, if A is to survive B and C .

Then, if B be supposed to die first, and C second; the value of A 's interest will (by quest. 46.) be

$$\frac{P - P}{n} + P - \frac{nn}{6rm} \times \frac{1}{2t} \times P;$$

And, if C be supposed to die first, and B second; then the value of A 's interest will (by quest. 47.) be

$$\frac{P - P}{m} + P - \frac{n}{6r} \times \frac{n}{2tm} \times P.$$

124 MATHEMATICAL

Therefore, their sum

$$\left. \begin{aligned} & \frac{a^2 - a^2}{n} + \frac{m^2 - m^2}{m} \\ & + \frac{a^2 - \frac{nn}{6rm}}{6rm} \times \frac{1}{2t} + \frac{m^2 - \frac{nn}{6r}}{6r} \times \frac{n}{2tm} \end{aligned} \right\} \times P$$

or) $\left. \begin{aligned} & \frac{a^2 - a^2}{n} + \frac{m^2 - m^2}{m} \\ & + \frac{a^2 + m^2 - \frac{2n}{6r} \times \frac{n}{m} \times \frac{1}{2t}}{6r} \end{aligned} \right\} \times P \text{ will be the}$

value required.

EXAMPLE,

C (aged 66) is possessed of an estate of 1*l.* per annum, in which his wife B (aged 54) is jointured; which estate will descend to his son A (aged 43) if he survives them both; what is the present value of A's interest in that estate?

Here $a^2 = 10,838$; $a^2 = 9,891$; $m^2 = 12,578$; $m^2 = 7,673$; $n = 20$; $m = 32$; $t = 43$; $P = 25$,

and $\frac{a^2}{6r} \times \frac{n}{m} = 0,789$ (by exam. case 2.)

$$\text{Then } \frac{10,838 - 9,891}{20} = 0,04735; \quad \frac{12,578 - 7,673}{32} = 0,15328;$$

$$\text{And } \frac{9,891 + 0,789}{2 \times 43} = 0,12419;$$

Therefore $(0,04735 + 0,15328 + 0,12419 \times 25 =)$
 $\$1,1205$ will be the value required.

C O R O L L

If the survivor and the younger of the persons, to be survived, are of equal ages; then $a^2 = a^2$, $m^2 = m^2$, and $t = n$; therefore

(39 —

$$\left(\frac{P - R}{m} + R + R - \frac{2n}{6r} \times \frac{n}{m} \times \frac{1}{2m} \times P = \right)$$

$$R - R \times 2 + R + R - \frac{2n}{6r} \times \frac{n}{m} \times \frac{P}{2m}$$

(as in corol. 1. case 2.) will be the value required.

COROL. II.

If the two persons, to be survived, are of equal ages; then $m = n$, $mP = nP$, and $nR = R$; therefore

$$\left(\frac{P - R \times 2}{n} + R + R - \frac{2n}{6r} \times \frac{1}{2t} \times P \text{ or} \right)$$

$\frac{P - R \times 2}{n} + R - \frac{n}{6r} \times \frac{1}{t} \times P$ will be the value required.

QUESTION XLIX.

A person who is intituled to an absolute estate, in fee simple, if he survives the present possessor, grants the reversion of an annuity of 1*l*. per annum, to be secured thereon, to a third person, for the remainder of his (the third person's) life after the decease of the present possessor; but so, that if the granter doth not live to come into possession, the annuity is not to take place: what is the present value of that reversion, so depending on the survivorship?

SOLUTION.

Let *A*, *B*, and *C*, represent the names of the youngest, second, and eldest, of the three persons; and let *s*, *m*, and *n*, be their respective complements of life.

G 3

CASE

CASE I.

If the possessor, be the youngest; the expectant, second; and the annuitant, the eldest, of the three persons.

Then the annual probabilities of the expectant's coming into possession will (by quest. 21) be $\frac{2m-1}{2m} \times \frac{1}{t}$, $\frac{2m-3}{2m} \times \frac{1}{t}$, $\frac{2m-5}{2m} \times \frac{1}{t}$, &c. that is $\frac{1}{t} - \frac{1}{2mt}$, $\frac{1}{t} - \frac{3}{2mt}$, $\frac{1}{t} - \frac{5}{2mt}$, &c.

Now, if the survivorship should take place in the first year, the annuitant will be entitled to the annuity for his whole life, viz. to $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} + \frac{2n-5}{2nr^3} (n)$;

Therefore if, $\frac{1}{t} - \frac{1}{2mt}$, the probability of the survivorship's taking place therein, be multiplied thereby; the product, viz. $\frac{1}{t} - \frac{1}{2mt} \times \frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} + \frac{2n-5}{2nr^3}$ (n), will be the annuitant's expectation for that year.

But, if the survivorship fails in the first, and takes place in the second year, then the annuitant will be entitled to the whole annuity for his life, except the first annual payment (which he will not receive, because the granter will not be then in possession) viz. $\frac{2n-3}{2nr^2} +$

$\frac{2n-5}{2nr^3} + \frac{2n-7}{2nr^4} (n-1)$; which, being multiplied by,

$\frac{1}{t} - \frac{3}{2mt}$, the probability of the survivorship's, so, taking place, will produce,

$\frac{1}{t} - \frac{3}{2mt} \times \frac{2n-3}{2nr^2} + \frac{2n-5}{2nr^3} + \frac{2n-7}{2nr^4} (n-1)$, the

annuitant's expectation for the second year.

In

In like manner, it may be shewn that the annuitants expectation, for the third year, will be

$$\frac{1}{i} - \frac{5}{2mi} \times \frac{2n-5}{2nr^3} + \frac{2n-7}{2nr^4} + \frac{2n-9}{2nr^5} (n-2), \&c.$$

- Now if these expectations be expanded by multiplication, and the affirmative terms, in each product, be separated from the negative, they will appear as below.

$$\begin{aligned} & + \frac{2n-1}{2nr} \times \frac{1}{i} + \frac{2n-3}{2nr^3} \times \frac{1}{i} + \frac{2n-5}{2nr^5} \times \frac{1}{i} + \frac{2n-7}{2nr^7} \times \frac{1}{i} \&c. \\ & + \frac{2n-3}{2nr^3} \times \frac{1}{i} + \frac{2n-5}{2nr^5} \times \frac{1}{i} + \frac{2n-7}{2nr^7} \times \frac{1}{i} \&c. \\ & + \frac{2n-5}{2nr^5} \times \frac{1}{i} + \frac{2n-7}{2nr^7} \times \frac{1}{i} \&c. \\ & + \frac{2n-7}{2nr^7} \times \frac{1}{i} \&c. \\ & - \frac{2n-1}{2nr} \times \frac{1}{2mi} - \frac{2n-3}{2nr^3} \times \frac{1}{2mi} - \frac{2n-5}{2nr^5} \times \frac{1}{2mi} - \frac{2n-7}{2nr^7} \times \frac{1}{2mi} \&c. \\ & - \frac{2n-3}{2nr^3} \times \frac{3}{2mi} - \frac{2n-5}{2nr^5} \times \frac{3}{2mi} - \frac{2n-7}{2nr^7} \times \frac{3}{2mi} \&c. \\ & - \frac{2n-5}{2nr^5} \times \frac{5}{2mi} - \frac{2n-7}{2nr^7} \times \frac{5}{2mi} \&c. \\ & - \frac{2n-7}{2nr^7} \times \frac{7}{2mi} \&c. \end{aligned}$$

G 4

Where

128 MATHEMATICAL

Where the affirmative serieses, being added, produce the series, $\frac{2n-1}{2nr} \times \frac{1}{t} + \frac{2n-3}{2nr^2} \times \frac{2}{t} + \frac{2n-5}{2nr^3} \times \frac{3}{t} + \frac{2n-7}{2nr^4} \times \frac{4}{t} (n);$

And the negative serieses, being added, produce the series $\{- \frac{2n-1}{2nr} \times \frac{1}{2mt} - \frac{2n-3}{2nr^2} \times \frac{4}{2mt} - \frac{2n-5}{2nr^3} \times \frac{9}{2mt} - \frac{2n-7}{2nr^4} \times \frac{16}{2mt} (n);$

or) $\frac{2n-1}{2nr} \times \frac{1}{2m} \times \frac{1}{t} - \frac{2n-3}{2nr^2} \times \frac{2}{2m} \times \frac{2}{t} - \frac{2n-5}{2nr^3} \times \frac{3}{2m} \times \frac{3}{t} - \frac{2n-7}{2nr^4} \times \frac{4}{2m} \times \frac{4}{t} (n);$

The first, of which may be summed, by quest. 21. vol. 2, and the latter, by quest. 22. vol. 2.

Now first, in order to find the sum of the affirmative series; It is known that the sum of $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} (n)$ is $\frac{1}{nr}$ and its common difference $+$ $\frac{1}{nr}$; and the sum of

$\frac{1}{t} + \frac{2}{t} (n)$ is $\frac{nn+1}{2t}$, and its common difference $- \frac{1}{t}$;

Whence (by quest. 21. vol. 2.) the sum of n terms of the series of products will be

$$\left(\frac{n \times \frac{nn+1}{2t}}{n} - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{nr} \text{ or } \right) \frac{n+1 \times n}{2t} - \frac{n+1 \times n-1}{2t \times nr};$$

That is $\frac{n-1}{nr} \times \frac{n+1}{2t}$; will be the sum of the affirmative series. Secondly,

Secondly, to find the sum of the negative series;

The sum of $\frac{2n-1}{2nr} + \frac{2n-3}{2nr^2} (n)$ is $\frac{1}{2}$, its first term

$\frac{2n-1}{2nr}$, and its common difference $\frac{1}{nr}$;

The sum of $\frac{1}{t} + \frac{2}{t} (n)$ is $\frac{nn+n}{2t}$, its first term $\frac{1}{t}$,

and its common difference $-\frac{1}{t}$;

The sum of $\frac{1}{2m} + \frac{2}{2m} (n)$ is $\frac{nn+n}{4m}$, its first term

$\frac{1}{2m}$, and its common difference $-\frac{1}{2m}$;

Whence the sum of the series of products will (by qu.

22. vol. 2.) be $\left(\frac{1}{2} \times \frac{nn+n}{2t} \times \frac{nn+n}{4m} \right.$

$+ \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2} \times \frac{2n-1}{4nmtr} - \frac{1}{2nmtr} - \frac{1}{2nmtr}$

$- \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2} \times \frac{1}{2nmtr} \text{ or } \left. \right)$

$\frac{1}{2} \times \frac{n+1}{2t} \times \frac{n+1}{4m} + \frac{n+1 \cdot n-1}{2 \cdot 6r} \times \frac{2n-5}{4mt}$

$\left(- \frac{n+1 \cdot n-1^2}{8mt \times 2r} \right);$

That is $\frac{1}{2} \times \frac{n+1}{2t} \times \frac{n+1}{4m} + \frac{n+1 \cdot n-1}{8mt} \times \frac{2n-5}{6r}$

$\left(- \frac{n+1 \cdot n-1}{8mt} \times \frac{3n-3}{6r} \right).$

But $\frac{2n-5}{6r} - \frac{3n-3}{6r} = -\frac{n+2}{6r}$; whence we shall

have $(P \times \frac{n+1}{2t} \times \frac{n+1}{4m} - \frac{n+1 \cdot n-1}{8mt}) \times \frac{n+2}{6r}$

Or) $P \times \frac{n+1 \cdot n+1}{8mt} - \frac{n+1 \cdot n+2}{8mt} \times \frac{n-1}{6r}$:

Now, if we write, $\frac{n+1 \times n+1}{8mt} \times \frac{n-1}{6r}$, for

$\frac{n+1 \times n+2}{8mt} \times \frac{n-1}{6r}$ (which differ by

$\frac{n+1 \times n-1}{48mtr}$ which is less than $\frac{1}{48r}$) the above ex-

pression will become $(P \times \frac{n+1}{8mt} - \frac{n+1}{8mt} \times \frac{n-1}{6r}$

or) $P - \frac{n-1}{6r} \times \frac{n+1}{8mt}$; which is the sum of the negative series.

Now, if from, $P - \frac{n-1}{6r} \times \frac{n+1}{2t}$, the sum of the affirmative series, we take, $P - \frac{n-1}{6r} \times \frac{n+1}{8mt}$, the sum of the negative series; then, the remainder

$(P - \frac{n-1}{6r} \times \frac{n+1}{2t} - \frac{n+1}{8mt})$

or) $P - \frac{n-1}{6r} \times 1 - \frac{n+1}{4m} \times \frac{n+1}{2t}$ will be the value of the annuitants whole expectation; that is, of the reversion depending on the survivorship.

EXAMPLE.

B (aged 54) who is heir to an estate, upon the death of his niece A (aged 43) if he be then alive, grants an annuity

ity of 1*l.* per annum to C (aged 66) for his life, secured on, and issuable out of the estate, to commence at the time, when he (B) enters into possession thereof; what present money ought C to pay, for the same?

$$\begin{aligned} \text{Here } R &= 7,673; \frac{n-1}{6r} = 3,045; \frac{n+1}{4m} = \left(\frac{21}{4 \cdot 32} \right. \\ &= \left. \frac{21}{128}; 1 - \frac{21}{128} = \left(\frac{128-21}{128} = \right) \frac{107}{128}; \right. \\ \frac{n+1}{2t} &= \left(\frac{21}{2 \cdot 43} = \right) \frac{21}{86}; \text{ and } 7,673 - 3,045 = \\ &4,628. \end{aligned}$$

Then $\left(\frac{4,628 \times 107 \times 21}{128 \times 86} = \right) 0,945$ will be the value required.

COROL. I.

If the possessor and expectant are of equal ages; then $t=m$; and the value of the annuitant's expectation will

$$\text{become, } R - \frac{n-1}{6r} \times 1 - \frac{n+1}{4m} \times \frac{n+1}{2m}.$$

COROL. II.

If the expectant and annuitant are of equal ages; then $m=n$; and the value required will be

$$\begin{aligned} &\left(R - \frac{n-1}{6r} \times 1 - \frac{n+1}{4n} \times \frac{n+1}{2t} \text{ or } \right) \\ &R - \frac{n-1}{6r} \times \frac{3n-1}{4n} \times \frac{n+1}{2t} \end{aligned}$$

COROL. III.

If the three persons be of equal ages; then,

$$P = \frac{n-1}{6r} \times \frac{3n-1}{4n} \times \frac{n+1}{2n}$$
, will be the value required.

CASE II.

If the expectant, be the youngest; the possessor, the second; and the annuitant, the eldest, of the three persons.

Then (by writing m for t , and t for m , in the solution of the last case) we shall have,

$$P = \frac{n-1}{6r} \times 1 - \frac{n+1}{4t} \times \frac{n+1}{2m}$$
, for the value required.

COROL. I.

Hence if $t = m$, the value will become,

$$P = \frac{n-1}{6r} \times 1 - \frac{n+1}{4m} \times \frac{n+1}{2m}$$
, as in corol. 1. case 1.

COROL. II.

If $m = n$; then
$$P = \frac{n-1}{6r} \times 1 - \frac{n+1}{4t} \times \frac{n+1}{2n}$$
 will be the value required.

CASE III.

If the possessor, be the youngest; the annuitant, the second; and the expectant, the eldest, of the three persons.

Then,

Then, the annual probabilities of the expectant's coming into possession will be $\frac{2n-1}{2n} \times \frac{1}{t}$, $\frac{2n-3}{2n} \times \frac{1}{t}$,

&c, continued only to n terms; which probabilities may (as in the last case) be wrote $\frac{1}{t} - \frac{1}{2nt}$, $\frac{1}{t} - \frac{3}{2nt}$ &c.

and consequently, if the survivorship takes place in the first, second, third, &c. year the annuitant's expectations will be,

$$\frac{1}{t} - \frac{1}{2nt} \times \frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3} (m),$$

$$\frac{1}{t} - \frac{3}{2nt} \times \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3} + \frac{2m-7}{2mr^4} (m-1),$$

$$\frac{1}{t} - \frac{5}{2nt} \times \frac{2m-5}{2mr^3} + \frac{2m-7}{2mr^4} + \frac{2m-9}{2mr^5} (m-2), \&c.$$

Of which serieses the number will be only n ; because the probability of the expectant's surviving the possessor ceases, at the end of n years.

Let now these serieses be supposed to be expanded by multiplication, and ranged as in the first case; and, let us examine the consequence of expounding the symbol, by the numbers 2, 3, and 4, respectively.

If $n = 2$; then, only, the first and second products will be to be added together; and the result will be represented by

$$\begin{aligned} & + \frac{2m-1}{2mr} \times \frac{1}{t} + \frac{2m-3}{2mr^2} \times \frac{2}{t} + \frac{2m-5}{2mr^3} \times \frac{2}{t} \\ & + \frac{2m-7}{2mr^4} \times \frac{2}{t} (m), - \frac{2m-1}{2mr} \times \frac{1}{2mt} - \frac{2m-3}{2mr^2} \times \frac{4}{2nt} \\ & - \frac{2m-5}{2mr^3} \times \frac{4}{2nt} - \frac{2m-7}{2mr^4} \times \frac{4}{2nt} (m). \end{aligned}$$

Where the terms following the 2 first, in each series, consist of such expressions as are found in the value of the single life of the annuitant, multiplied into the constant factors,

$$\frac{2}{t}$$

134 MATHEMATICAL

$\frac{2}{t}$ and $\frac{4}{2nt}$; which (because they arise from the supposition of the equality of 2 and n) may be wrote $\frac{n}{t}$ and

$\left(\frac{nn}{2nt} \text{ or } \right) \frac{n}{2t}$; and consequently (in this case) the negative series will, after the second term, be just half the affirmative.

Again, if $n = 3$; then three of the serieses, produced by the multiplications, will be to be added together; and the result will be represented by

$$+ \frac{2m-1}{2mr} \times \frac{1}{t} + \frac{2m-3}{2mr^2} \times \frac{2}{t} + \frac{2m-5}{2mr^3} \times \frac{3}{t} \\ \left(+ \frac{2m-7}{2mr^4} \times \frac{3}{t} + \frac{2m-9}{2mr^5} \times \frac{3}{t} (m), \right. \\ \left. - \frac{2m-1}{2mr} \times \frac{1}{2nt} - \frac{2m-3}{2mr^2} \times \frac{4}{2nt} - \frac{2m-5}{2mr^3} \times \frac{9}{2nt} \right. \\ \left. \left(- \frac{2m-7}{2mr^4} \times \frac{9}{2nt} - \frac{2m-9}{2mr^5} \times \frac{9}{2nt} (m); \right) \right.$$

wherein, after the three first terms, the factors (which are multiplied into the terms of the single life of the annuitant) become invariably $\frac{3}{t}$ and $\frac{9}{2nt}$; which (because

$n = 3$) may be wrote $\frac{n}{t}$ and $\left(\frac{nn}{2nt} \text{ or } \right) \frac{n}{2t}$ as before.

And, if we proceed to make $n = 4$, we shall find the same things occur after four terms; and consequently the value of the annuity required will be represented by

$$\frac{2m-1}{2mr} \times \frac{1}{t} + \frac{2m-3}{2mr^2} \times \frac{2}{t} + \frac{2m-5}{2mr^3} \times \frac{3}{t} (n) \\ \left(+ \frac{2m-2n-1}{2mr^{n+1}} \times \frac{n}{t} + \frac{2m-2n-3}{2mr^{n+2}} \times \frac{n}{t} (m-n); \right. \\ \left. - 2m-1 \right.$$

$$\begin{aligned}
 & - \frac{2m-1}{2mr} \times \frac{1}{2nt} - \frac{2m-3}{2mr^2} \times \frac{1}{2nt} \\
 & - \frac{2m-5}{2mr^3} \times \frac{1}{2nt} (n) - \frac{2m-2n-1}{2mr^{n+1}} \times \frac{nn}{2tn} \\
 & \left(- \frac{2m-2n-3}{2mr^{n+2}} \times \frac{nn}{2tn} (m-n) \right)
 \end{aligned}$$

Now, because $\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} + \frac{2m-5}{2mr^3} (n) =$

$\frac{2m-2n-1}{2mr^{n+1}} + \frac{2m-2n-3}{2mr^{n+2}} (m-n)$

$= 0$; and, because $\frac{n}{t} - \frac{n}{2t} = \frac{n}{2t}$; therefore, the difference of those parts of the affirmative and negative series, which have invariable factors, will be

$0 - 0 \times \frac{n}{2t}$

It remains therefore, only, to find the sum of those n terms of each series, which have variable factors.

First, in order to find the sum of, $\frac{2m-1}{2mr} \times \frac{1}{t}$
 $+ \frac{2m-3}{2mr^2} \times \frac{2}{t} (n)$, it is known, that $\frac{2m-1}{2mr} +$
 $\frac{2m-3}{2mr^2} (n) = 0$; the common difference thereof being $\frac{1}{mr}$;

Also $\frac{1}{t} + \frac{2}{t} (n) = \frac{nn+n}{2t}$, and the common difference

$-\frac{1}{t}$; therefore the sum of n terms of the series of products will (by quest. 21. vol. 2.) be

(0)

126 MATHEMATICAL

$$= \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2} \times \frac{1}{n(n+1)}$$

$$= \frac{1}{2} \times \frac{n+1}{n(n+1)} = \frac{1}{2n}$$

$$= \frac{1}{2n} \text{ will be the sum of the } n \text{ terms.}$$

$$\text{The sum of the series, } \frac{1}{2n} \times \frac{1}{2n}$$

$$= \frac{1}{4n^2}$$

$$\text{The sum of the series } \frac{1}{2n} + \frac{1}{2n} \text{ is } \frac{1}{n}; \text{ its first}$$

$$\text{term is } \frac{1}{2n}, \text{ and common difference } \frac{1}{2n}.$$

$$\text{The sum of the series } \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} \text{ is } \frac{3}{2n}; \text{ its first term } \frac{1}{2n},$$

$$\text{and common difference } \frac{1}{2n}.$$

$$\text{The sum of the series } \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} \text{ is } \frac{2}{n}; \text{ its first term } \frac{1}{2n},$$

$$\text{and common difference } \frac{1}{2n}.$$

$$\text{The sum of the series } \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} + \frac{1}{2n} \text{ is } \frac{5}{2n}; \text{ its first term } \frac{1}{2n},$$

$$\text{and common difference } \frac{1}{2n}.$$

$$= \left(\frac{n+1}{4n} \times \frac{n+1}{4n} \times \frac{n+1}{2t} + \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \right) \text{ or } \\ \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}, \text{ will be the sum of the ne-}$$

series, which have variable factors, will be

$$= \frac{n-1}{6rm} \times \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \text{ or}$$

$$= \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}; \text{ that is,}$$

$\frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}$, will be the difference of the factors.

difference of those, which have invariable

$$= \frac{n+1}{2t} \times \frac{n+1}{4n} + \frac{n+1}{8mt} \times \frac{2m-5}{6r} \\ \left(- \frac{n+1}{8mt} \times \frac{3n-3}{6r} \right);$$

136 MATHEMATICAL

$$= \frac{n+1}{2t} \times \frac{1}{n} - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{mtr}$$

or) $\frac{n+1}{2t} - \frac{n \cdot n-1}{6rm} \times \frac{n+1}{2t}$; that is

$\frac{n+1}{2t} - \frac{n \cdot n-1}{6rm} \times \frac{n+1}{2t}$ will be the sum of the affirmative series.

Again to find the sum of the series, $\frac{2m-1}{2mr} \times \frac{1}{2nt}$
 $+ \frac{2m-3}{2mr^2} \times \frac{4}{2nt} (n)$;

The sum of $\frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} (n)$ is $\frac{n+1}{2t}$; its first term $\frac{2m-1}{2mr}$; and common difference $\frac{1}{mr}$:

The sum of $\frac{1}{t} + \frac{2}{t} (n)$ is $\frac{n+1}{2t}$; its first term $\frac{1}{t}$; and common difference $-\frac{1}{t}$:

The sum of $\frac{1}{2n} + \frac{2}{2n} (n)$ is $\left(\frac{n+1}{4n} =\right) \frac{n+1}{4}$; its first term $\frac{1}{2n}$; and common difference $-\frac{1}{2n}$:

Whence, the sum of the series of products will (by quest. 22. vol. 2.) be $\left(\frac{n+1}{2t} \times \frac{n+1}{4} \times \frac{1}{nn}\right.$

$$+ \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{2m-1}{4mnt} - \frac{1}{2mnt} - \frac{1}{2mnt} - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2} \times \frac{1}{2mnt} \text{ or } \left. \right)$$

$$\begin{aligned}
 & \times \frac{n+1}{2t} \times \frac{n+1}{4n} + \frac{n+1 \cdot n-1}{8mt} \times \frac{2m-5}{6r} \\
 & \left(- \frac{n+1 \cdot n-1}{8mt} \times \frac{3n-3}{6r} \right)
 \end{aligned}$$

That is $(\frac{n+1}{4n} \times \frac{n+1}{2t} + \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \text{ or})$
 $\frac{n+1}{4n} \times \frac{n-1}{6r} + \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}$, will be the sum of the ne-

gative series.

And the difference of the two series, which have variable factors, will be

$$\begin{aligned}
 & (\frac{n-1}{6rm} - \frac{n-1}{6rm} \times \frac{n+1}{4n} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \text{ or}) \\
 & 1 - \frac{n+1}{4n} \times \frac{n-1}{6r} - \frac{n-1}{6r} \times \frac{n}{m} + \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}; \text{ that is,} \\
 & \frac{3n-1}{4n} \times \frac{n-1}{6r} - \frac{n-1}{6r} \times \frac{2m+n-2}{4m} \times \frac{n+1}{2t}, \text{ will be the difference of the} \\
 & \text{two serieses, which have variable factors.}
 \end{aligned}$$

To which, if $\frac{n-1}{6r} - \frac{n}{m} \times \frac{n}{2t}$, the difference of those, which have invariable factors, be added, the sum

$$\left(\frac{3n-1}{4^n} \times \frac{n-1}{6r} \times \frac{2m+n-2}{4^m} \times \frac{n+1}{2^t} + \frac{n-1}{6r} \times \frac{n}{2^t} \right) \text{ or } \frac{3n-1}{4^n} \times \frac{n-1}{6r} \times \frac{2m+n-2}{4^m} \times \frac{n+1}{2^t} + \frac{n-1}{6r} \times \frac{n}{2^t} \times \frac{1}{2^t}$$

will be the value of the annuitant's expectation.

EXAMPLE.

A (aged 43) is possessed of an estate, which on her decease will devolve to her uncle C (aged 66) if he be then alive; what is the present worth of the reversion of an annuity (to be secured on that estate) for the remainder of the life of B (aged 54) to commence when C becomes possessed of the estate?

$$\text{Here } 10,757; \frac{n-1}{6r} = 9,891; \frac{n-1}{6r} = 3,045; m = 32; n = 20; t = 43;$$

$$\frac{3n-1}{4^n} = \frac{59}{80}; n+1 = 21; \text{ and } \frac{2m+n-2}{4^m} = \left(\frac{64+20-2}{4 \cdot 32} \right) = \frac{82}{128};$$

$$\text{Then } 9,891 \times \frac{59}{80} = 7,295; 3,045 \times \frac{82}{128} = 1,951; 7,295 - 1,951 = 5,344;$$

And

And $5,344 \times 21 = 112,224$;

Again $10,757 - 9,891 = 0,866$; and $0,866 \times 20 = 17,320$;

Now $112,224 + 17,320 = 129,544$; and $\frac{129,544}{2.43} = 1,506$ the value required.

COROL. I.

If the possessor and annuitant are of equal ages, then $t = m$, and the value of the reversion will become

$$\left. \begin{aligned} & \frac{3n-1}{4^n} \cdot \frac{n-1}{6r} \times \frac{2n+1-2}{4^n} \times n+1 \\ & + \frac{n-1}{6r} \times n \end{aligned} \right\} \times \frac{1}{2^n}$$

COROL. II.

If the expectant and annuitant are of equal ages; then $m = n$, and $\frac{n-1}{6r} = \frac{n-1}{6r}$; whence the value re-

quired will be $\frac{3n-1}{4^n} \cdot \frac{n-1}{6r} \times \frac{3n-2}{4^n} \times \frac{n+1}{2^n}$;

That is (if we write $3n-1$, for $3n-2$)

$\frac{n-1}{6r} \times \frac{3n-1}{4^n} \times \frac{n+1}{2^n}$ will be the value required, as in corol. 2. case 1.

CASE IV.

If the possessor, be the eldest; the annuitant, the second; and the expectant the youngest, of the three persons.

Then the annual probabilities of the expectant's coming into possession will be $\frac{2t-1}{2t} \times \frac{1}{n}$; $\frac{2t-3}{2t} \times \frac{1}{n}$ &c; which

which may (as before) be otherwise wrote $\frac{1}{n} - \frac{1}{2nt}$;

$\frac{1}{n} - \frac{3}{2nt}$ &c. and, the process being continued in the foregoing manner, we shall find that the value of the reversion of the annuity required will be expressed, by the difference of the two following series,

$$\begin{aligned} & \frac{2m-1}{2mr} \times \frac{1}{n} + \frac{2m-3}{2mr^2} \times \frac{2}{n} + \frac{2m-5}{2mr^3} \times \frac{3}{n} (n) \\ & + \frac{2m-2n-1}{2mr^{n+1}} \times \frac{n}{n} + \frac{2m-2n-3}{2mr^{n+2}} \times \frac{n}{n} (m-n); \text{ and} \\ & - \frac{2m-1}{2mr} \times \frac{1}{2nt} - \frac{2m-3}{2mr^2} \times \frac{4}{2nt} - \frac{2m-5}{2mr^3} \times \frac{9}{2nt} (n) \\ & - \frac{2m-2n-1}{2mr^{n+1}} \times \frac{nn}{2nt} - \frac{2m-2n-3}{2mr^{n+2}} \times \frac{nn}{2nt} (m-n); \end{aligned}$$

where (because $\frac{n}{n} = 1$, and $\frac{nn}{2nt} = \frac{n}{2t}$) the difference of those parts of the two series, which have invariable factors, will be $\frac{1}{n} - \frac{n}{2t} \times 1 - \frac{n}{2t}$.

In the affirmative series, having variable factors, viz.

$$\frac{2m-1}{2mr} \times \frac{1}{n} + \frac{2m-3}{2mr^2} \times \frac{2}{n} (n); \frac{2m-1}{2mr} + \frac{2m-3}{2mr^2} (n)$$

= $\frac{1}{mr}$, and the common difference = $\frac{1}{n}$; $\frac{1}{n} + \frac{2}{n} (n) = \left(\frac{nn+n}{2n} = \right) \frac{n+1}{2}$, and the common difference is $-\frac{1}{n}$; therefore the sum of the series of products

will be $\left(\frac{n+1}{2} \times \frac{n+1}{2} \times \frac{1}{n} - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \frac{1}{mr} \right)$
or)

$$\text{or) } \frac{n+1}{2n} - \frac{n+1 \cdot n-1}{2m \cdot 6r}$$

The negative series, having variable factors, is the same with that before exhibited; and consequently its sum will be

$$\frac{n+1}{4n} + \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t}$$

This, being subtracted, from the above found sum of the affirmative series, having variable factors, will leave

$$\left(\frac{n+1}{2n} - \frac{n+1 \cdot n-1}{2m \cdot 6r} - \right.$$

$$\left. \frac{n+1}{4n} + \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \text{ or) } \right)$$

$$\frac{n+1}{2n} - \frac{n+1 \cdot n-1}{2m \cdot 6r} - \frac{n+1}{2n} \times \frac{n+1}{4t}$$

$$\left(- \frac{n-1}{6r} \times \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} ; \right.$$

$$\text{That is } \left(\frac{n+1}{4t} \times \frac{n+1}{2n} \right.$$

$$\left. \left(- \frac{n-1}{6r} \times \frac{n+1}{2m} + \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} \right) \right)$$

$$\text{Or) } \frac{n+1}{4t} \times \frac{n+1}{2n}$$

$$\left(- \frac{n-1}{6r} \times \frac{n+1}{2m} + \frac{2m-3n-2}{4m} \times \frac{n+1}{2t} ; \right.$$

$$\text{That is, } \frac{n+1}{4t} \times \frac{n+1}{2n}$$

$$\left(- \frac{n-1}{6r} \times 1 + \frac{2m-3n-2}{4t} \times \frac{n+1}{2m} \right.$$

will be the difference of the two series, which have variable factors.

To

144 MATHEMATICAL

Then the annual probabilities of the expectant's coming into possession will be $\left(\frac{2n-1}{2n} \times \frac{1}{m}, \frac{2n-3}{2n} \times \frac{1}{m}, \dots\right.$
 &c. or) $\frac{1}{m} - \frac{1}{2nm}, \frac{1}{m} - \frac{3}{3nm}, \dots$ and these, being multiplied into the proper terms of the series, $\frac{2t-1}{2tr} +$
 $\frac{2t-3}{2tr^2}$ &c. expressing the value of the annuitant's life, will produce the two following series,

$$\begin{aligned} &\frac{2t-1}{2tr} \times \frac{1}{m} + \frac{2t-3}{2tr^2} \times \frac{2}{m} + \frac{2t-5}{2tr^3} \times \frac{3}{m} (n) \\ &+ \frac{2t-2n-1}{2tr^{n+1}} \times \frac{n}{m} + \frac{2t-2n-3}{2tr^{n+2}} \times \frac{n}{m} (t-n); \text{ and} \\ &= \frac{2t-1}{2tr} \times \frac{1}{2nm} - \frac{2t-3}{2tr^2} \times \frac{4}{2nm} - \frac{2t-5}{2tr^3} \times \frac{9}{2nm} (n) \\ &- \frac{2t-2n-1}{2tr^{n+1}} \times \frac{nn}{2nm} - \frac{2t-2n-3}{2tr^{n+2}} \times \frac{nn}{2nm} (t-n); \end{aligned}$$

and their difference will be the value of the annuitant's expectation.

If these series be compared, with those in the third case; it will appear, that if we write, $\frac{1}{m}$ for $\frac{1}{n}$, $\frac{1}{n}$ for $\frac{1}{m}$, t for m , and m for t , in the solution of that case; the solution of this case will be, thereby, obtained: therefore

$$\left. \begin{aligned} &\frac{3n-1}{4n} \times \frac{1}{m} - \frac{n-1}{6r} \times \frac{2t+n-2}{4t} \times \frac{n+1}{m} \\ &+ \frac{1}{m} - \frac{1}{m} \times n \end{aligned} \right\} \times \frac{1}{2m}$$

will be the value of the annuitant's expectation.

EXAM-

EXAMPLE.

C (aged 66) who will come into possession of an estate, at the decease of B (aged 54) if he (C) be then alive, would grant an annuity (secured on that estate) to A (aged 43) which is to commence, when C takes possession of the estate, and continue during the remainder of A's life: the present worth thereof is required.

Here $\ddot{a}_x = 12,920$; $\ddot{a}_y = 10,838$; $\frac{x-1}{6r} = 3,045$;
 $\frac{3x-1}{4x} = \frac{59}{80}$; $\frac{2t+x-2}{4t} = \left(\frac{86+20-2}{4 \times 43} = \frac{104}{4 \times 43} \right.$
 $\left. = \right) \frac{26}{43}$; $x+1 = 21$; $\ddot{a}_x - \ddot{a}_y = 2,082$; $x = 20$;
 and $m = 32$.

Then $12,920 \times \frac{59}{80} = 9,528$; $3,045 \times \frac{26}{43} = 1,841$;
 $9,528 - 1,841 = 7,687$; $7,687 \times 21 = 161,427$;
 $2,082 \times 20 = 41,640$; and $161,427 + 41,640 = 203,067$.

Whence $\left(\frac{203,067}{2 \times 32} = \right) 3,173$ will be the value required.

COROL. I.

If the annuitant and possessor be of equal ages; then $\ddot{a}_x = \ddot{a}_y$, $\ddot{a}_x = \ddot{a}_y$, and $t = m$: whence the value of the annuitant's expectation will become,

$$\left. \begin{aligned} & \frac{3x-1}{4x} \times \ddot{a}_x - \frac{x-1}{6r} \times \frac{2m+x-2}{4m} \times x+1 \\ & + \ddot{a}_x - \ddot{a}_x \times x \end{aligned} \right\} \times \frac{1}{2m}$$

as in Corol. 1. case 3.

C O R O L L A R Y . II.

If the possessor and expectant are of equal ages; then $m = n$; and the value required will become,

$$\left. \begin{aligned} & \frac{3n-1}{4n} \times \frac{1}{n} - \frac{n-1}{6n} \times \frac{2t+n-2}{4n} \times \frac{1}{n+1} \\ & + \frac{1}{n} - \frac{1}{n} \times n \end{aligned} \right\} \times \frac{1}{2n}$$

C A S E VI.

If the annuitant, be the youngest; the expectant, second; and the possessor, the eldest, of the three persons. Then the annual probabilities of the expectant's possessing the estate will be $\left(\frac{2m-1}{2m} \times \frac{1}{n}, \frac{2m-3}{2m} \times \frac{1}{n} \&c. \right.$

or) $\frac{1}{n} - \frac{1}{2nm}, \frac{1}{n} - \frac{3}{2nm}, \&c.$ which (multiplied into

the proper terms of the series, expressing the value of the annuitant's life) will produce the two following series,

$$\begin{aligned} & \frac{2t-1}{2tr} \times \frac{1}{n} + \frac{2t-3}{2tr} \times \frac{2}{n} + \left(\frac{2t-5}{2tr} \times \frac{3}{n} \right) (n) \\ & + \frac{2t-2n-1}{2tr^{n+1}} \times \frac{n}{n} + \frac{2t-2n-3}{2tr^{n+2}} \times \frac{n}{n} (t-n); \text{ and} \end{aligned}$$

$$\begin{aligned} & - \frac{2t-1}{2tr} \times \frac{1}{2nm} - \frac{2t-3}{2tr} \times \frac{4}{2nm} - \frac{2t-5}{2tr} \times \frac{9}{2nm} (n) \\ & - \frac{2t-2n-1}{2tr^{n+1}} \times \frac{nn}{2nm} - \frac{2t-2n-3}{2tr^{n+2}} \times \frac{nn}{2nm} (n) \end{aligned}$$

whose difference, will be the value of the annuitant's expectation.

If these series be compared with those, in the fourth case, it will appear, that if we write t for m , and m for t , in the solution of that case; then

then the result $\frac{n-1}{6r} \times 1 + \frac{2t-3n-2}{4m} \times \frac{n+1}{2t} + \frac{n-1}{6r} \times 1 - \frac{n}{2m}$

will be the value of the annuitant's expectation, in this case.

EXAMPLE.

B (aged 54) who will possess an estate, on the death of his brother *C* (aged 66) if he (*B*) be the survivor, would grant an annuity (secured on that estate) to *A* (aged 43) which is to commence when *B* comes into possession, and to continue during the remainder of *A*'s life: what is the present worth thereof?

$$\begin{aligned} \text{Here } \mathcal{F} &= 12,620; \mathcal{F}^n = 10,838; \frac{n+1}{4m} = \left(\frac{21}{4 \cdot 32} \right. \\ &= \left. \right) \frac{21}{128}; 1 - \frac{21}{128} = \frac{107}{128}; \frac{n+1}{2t} = \frac{21}{40}; \frac{n-1}{6r} \\ &= 3,045; \frac{2t-3n-2}{4m} = \left(\frac{86-60-2}{4 \cdot 32} = \frac{24}{4 \cdot 32} \right) \\ &= \frac{3}{16}; 1 + \frac{2t-3n-2}{4m} = \left(1 + \frac{3}{16} = \right) \frac{19}{16}; \frac{n+1}{2t} \\ &= \frac{21}{86}; \mathcal{F} - \mathcal{F}^n = 2,082; 1 - \frac{n}{2m} = \left(\frac{44}{64} = \right) \\ &\frac{11}{16}; \end{aligned}$$

$$\begin{aligned} \text{Then } 10,838 \times \frac{107}{128} \times \frac{21}{40} &= 4,756; 3,045 \times \frac{19}{16} \\ &= 0,883; 2,082 \times \frac{11}{16} = 1,431; \text{ and } (4,756 - 0,883 \\ &+ 1,431 =) 5,304 \text{ will be the value required.} \end{aligned}$$

COROL. I.

If the annuitant and expectant are of equal ages; then $n = m$, $r = 1$, and $t = m$; whence the value required will become, $n \times 1 - \frac{n+1}{4n} \times \frac{n+1}{2n}$

$$- \frac{n-1}{6r} \times \frac{6n-3n-2}{4n} \times \frac{n+1}{2n} + 1 - n \times 1 - \frac{n}{2n}$$

as in corol. 1. Case 4.

COROL. II.

If the expectant and possessor are of equal ages; then $m = n$; and the required value will become

$$\left(n \times \frac{3n-1}{4n} \times \frac{n+1}{2n} - \frac{n-1}{6r} \times 1 + \frac{2t-3n-2}{4n} \times \left(\frac{n+1}{2t} + 1 - n \times 1 - \frac{n}{2n} \right) \right)$$

Or)

$$n \times \frac{3n-1}{4n} \times \frac{n+1}{2n} - \frac{n-1}{6r} \times \frac{2t+n-2}{4n} \times \frac{n+1}{2t} + 1 - n \times \frac{1}{2};$$

That is

$$\left(n \times \frac{3n-1}{4n} \times \frac{n+1}{2n} - \frac{n-1}{6r} \times \frac{2t+n-2}{4t} \times \frac{n+1}{2n} + 1 - n \times \frac{1}{2} \right)$$

Or)

$$n \times \frac{3n-1}{4n} - \frac{n-1}{6r} \times \frac{2t+n-2}{4t} \times \frac{n+1}{2n} + 1 - n \times \frac{n}{2n};$$

There-

Therefore, the value required may be expressed by,

$$\left. \begin{aligned} & n^2 \times \frac{3n-1}{4n} - \frac{n-1}{6r} \times \frac{2t+n-2}{4t} \times n+1 \\ & + \frac{n}{2} - n^2 \times n \end{aligned} \right\} \times \frac{1}{2n},$$

as in corol. 2. case 5.

QUESTION L.

The respective ages of 4 persons, *A* the youngest; *B*, the second; *C*, the third; and *D*, the eldest, being given. It is required to determine the probability of their dying, in the following order; viz. that *A* and *B*, the two youngest, shall die before either of the two eldest; that *C*, the next younger, shall die next; and that *D*, the eldest, shall survive the other three: and it is farther required, to find the present value of *D*'s interest in an estate, of 1*l.* per ann. or a legacy of 25*l.* dependent on this order of survivorship's taking place?

SOLUTION.

Let the complements of *A*, *B*, *C*, and *D*, be severally denoted by *t*, *m*, *n*, and *v*; then (since $\frac{2t-1}{2t}$ and $\frac{2m-1}{2m}$ are the separate expectations of life, of *A*, and *B*, for the first year) $1 - \frac{2t-1}{2t}$, and $1 - \frac{2m-1}{2m}$, will be the separate expectations of their death, at the beginning, or early in that year; whence $(1 - \frac{2t-1}{2t}) \times (1 - \frac{2m-1}{2m})$ or $1 - \frac{2t-1}{2t} - \frac{2m-1}{2m} + \frac{2t-1}{2t} \times \frac{2m-1}{2m}$ will be the expectation, that both of them will die at, or about

H 3

that

150 MATHEMATICAL

that time; if therefore this expectation be multiplied by, $\frac{2v-1}{2v} \times \frac{1}{n}$, the probability of D's surviving C, for that year, then the product

$$(1 - \frac{2t-1}{2t} - \frac{2m-1}{2m} + \frac{2t-1}{2t} \times \frac{2m-1}{2m} \times \frac{2v-1}{2v} \times \frac{1}{n}$$

$$\text{or } \frac{2v-1}{2v} \times \frac{1}{n} - \frac{2t-1}{2t} \times \frac{2v-1}{2v} \times \frac{1}{n} -$$

$$\frac{2m-1}{2m} \times \frac{2v-1}{2v} \times \frac{1}{n} + \frac{2t-1}{2t} \times \frac{2m-1}{2m} \times \frac{2v-1}{2v} \times \frac{1}{n}$$

will be the probability of the proposed order of survivorship's taking place, in the first year: and (by argu-

$$\text{ing in the same manner) } \frac{2v-3}{2v} \times \frac{1}{n}$$

$$- \frac{2t-3}{2t} \times \frac{2v-3}{2v} \times \frac{1}{n} - \frac{2m-3}{2m} \times \frac{2v-3}{2v} \times \frac{1}{n}$$

$$+ \frac{2t-3}{2t} \times \frac{2m-3}{2m} \times \frac{2v-3}{2v} \times \frac{1}{n}, \frac{2v-5}{2v} \times \frac{1}{n}$$

$$- \frac{2t-5}{2t} \times \frac{2v-5}{2v} \times \frac{1}{n} - \frac{2m-5}{2m} \times \frac{2v-5}{2v} \times \frac{1}{n}$$

$$+ \frac{2t-5}{2t} \times \frac{2m-5}{2m} \times \frac{2v-5}{2v} \times \frac{1}{n}, \&c. \text{ will be the}$$

probabilities of the proposed order of survivorship's taking place, in the second, third, &c. years: where the

$$\text{sum of the first affirmative series } \frac{2v-1}{2v} \times \frac{1}{n} +$$

$$\frac{2v-3}{2v} \times \frac{1}{n} + \frac{2v-5}{2v} \times \frac{1}{n} (v) \text{ is the prob. of } D's \text{ surviv-}$$

ing C, by quest. 21. the sums of the 2 negative series viz.

$$\frac{2t-1}{2t} \times \frac{2v-1}{2v} \times \frac{1}{n} + \frac{2t-3}{2t} \times \frac{2v-3}{2v} \times \frac{1}{n} (v)$$

and

$$\text{and } \frac{2m-1}{2m} \times \frac{2v-1}{2v} \times \frac{1}{n} + \frac{2m-3}{2m} \times \frac{2v-3}{2v} \times \frac{1}{n} (v)$$

are severally, the probabilities, that both *A*, and *D*, will survive *C*, and that both *B*, and *D*, will survive *C*, by case 1. quest. 31. and the sum of the last affirma-

$$\text{tive series } \left(\frac{2t-1}{2t} \times \frac{2m-1}{2m} \times \frac{2v-1}{2v} \times \frac{1}{n} + \frac{2t-3}{2t} \times \frac{2m-3}{2m} \times \frac{2v-3}{2v} \times \frac{1}{n} (v) \right) \text{ is the pro-}$$

bability that all the three persons, *A*, *B*, and *D* will survive *C*, by case 1. quest. 33.

Hence, if we call (as before) the two persons, whose deaths are required to happen first in order, the possessors; the person, upon whose death the survivorship will take place, the expectant; and the fourth person, the survivor: Then, from the probability of the survivor's outliving the expectant, take the probabilities, that both the elder possessor and survivor, and also that both the younger possessor and survivor, shall, severally, outlive the expectant; then, to the remainder, add the probability that the two possessors and survivor shall, all three of them, outlive the expectant; so shall the result be the probability, that the two possessors shall both die before, either the expectant or survivor; and, that the expectant will, also, die before the survivor; which probability was required.

Then (by proceeding to subtract, and add, in the manner above directed) the probability required will appear to be $\frac{v^3}{12tm} \times \frac{1}{n}$.

Again, since, $\mathcal{P} - \frac{v}{6r} \times \frac{vv}{4mt}$, is the sum of the series of present worths, which corresponds with the series of probabilities, whose sum is $\frac{v^3}{12tm}$;

Therefore, the interest of D in an estate, or legacy, worth \mathcal{P} pounds, will be $(\mathcal{P} - \frac{v}{6r} \times \frac{vv}{4mt} \times \frac{\mathcal{P}}{n} =)$
 $\mathcal{P} - \frac{v}{6r} \times \frac{vv\mathcal{P}}{4mtn}$.

EXAMPLE.

If A (aged 43) and B (aged 54) are possessed of an estate of 1*l.* per annum; which is to descend to C (aged 66) if he survives them both, and will after that come into the possession of D (aged 76) if he be alive at the death of C ; required the probability that he (D) has of becoming possessed of the estate, and the present worth of his interest therein?

Here $v=10$; $n=20$; $m=32$; $t=43$; whence the probability of his becoming possessed of the estate will be

$$\left(\frac{10 \times 10 \times 10}{12 \times 43 \times 32 \times 20} = \right) 0,00303; \text{ again } \mathcal{P} - \frac{v}{6r} =$$

2,715, and $\mathcal{P} = 25$; whence, the present worth of his

interest in the estate will be $\left(\frac{2,715 \times 10 \times 10 \times 25}{4 \times 43 \times 32 \times 20} = \right)$

2,0617.

COROL. I.

If the two possessors, are of equal ages; then $t = m$; therefore the answers to the question will be

$$\frac{v^3}{12mn^2}, \text{ and } P = \frac{v}{6r} \times \frac{vvP}{4mn}$$

COROL. II.

If the elder possessor, and expectant, are of equal ages; then $m = n$; and the answers to the question will be

$$\frac{v^3}{12tn}, \text{ and } P = \frac{v}{6r} \times \frac{vvP}{4tn}$$

COROL. III.

If the expectant, and survivor, are of equal ages; then $s = v$; and the answers to the question will be

$$\frac{vv}{12tm}, \text{ and } P = \frac{v}{6r} \times \frac{vP}{4tm}$$

COROL. IV.

If the two possessors, and the expectant, are of equal ages; then $t = m = n$; therefore the answers to the question will be

$$\frac{v^3}{12n^3}, \text{ and } P = \frac{v}{6r} \times \frac{vvP}{4n^2}$$

COROL. V.

If the elder possessor, the expectant, and the survivor, are of equal ages; then $m = n = v$; and the answers to the question will be

$$\frac{v}{12t}, \text{ and } P = \frac{v}{6r} \times \frac{P}{4t}$$

COROL.

COROLL. VI.

If the four lives be of equal ages, then $t=m=n=v$; and the answers to the question will become $\frac{1}{12}$, and

$$P = \frac{v}{6r} \times \frac{P}{4v}$$

Note. As each of the following eleven questions, are of the same nature with this, and admit of several corollaries; they will, for the sake of brevity, be proposed by making use of the letters A, B, C , and D , instead of the names, and t, m, n , and v , instead of the complements of the youngest, second, third, and eldest; the solutions (so far as regard the probabilities of survivorship) will proceed according to the rule printed in Italics in page 151; and the present worths will be found, either directly from those results, or by a similar process, as the nature of the case may require.

QUESTION LI.

The probability, that A and C , considered as possessors, shall both die before, either B , or D ; and that B , considered as the expectant, shall die before D , the survivor: also, the present worth of D 's interest, in an estate, or legacy, worth P pounds, dependent on this order of survivorship; are required.

which probabilities (being subtracted, and added, according to the manner prescribed) will give, $\frac{v^3}{12nm}$, for the probability required; and consequently,

$\mathfrak{P} = \frac{v}{6r} \times \frac{vv\mathfrak{P}}{4mtu}$, for the present worth of D 's interest, as in the last question.

COROL.

These answers (being the same with those to the last question) all the varieties (occasioned, by supposing two, or three, of the given lives to be of the same ages) will have the same results, as in the corollaries to the last question; interchanging, only, the names of *possessor*, and *expectant*, in their application to the lives, denoted by B , and C .

QUESTION LII.

The probability, that B and C , considered as possessors, shall both die before, either A , or D ; and that A , considered as the expectant, shall die before D , the survivor; also the present worth of D 's interest, in an estate, or legacy, worth \mathfrak{P} pounds, dependent on this order of survivorship; are required.

SOLU-

Here, the probability, that

will survive A , is

$$\frac{21}{33} \times \frac{1}{2} + \frac{31}{33} \times \frac{1}{2} + \frac{33}{33} \times \frac{1}{2}$$

by question

$$\frac{21}{33} \times \frac{1}{2} + \frac{31}{33} \times \frac{1}{2} + \frac{33}{33} \times \frac{1}{2}$$

SOLUTION

which

which probabilities (being subtracted, and added, according to the manner prescribed) will again, give, $\frac{v^3}{12amt}$, for the probability required; and consequently,

$\mathfrak{P} - \frac{v}{6r} \times \frac{vv\mathfrak{P}}{4amt}$, for the present worth of D 's interest, as in the two last questions.

COROL.

Hence the results of all the varieties (occasioned by supposing two, or three, of the given lives to be of equal ages) will be the same as in the corollaries to quest. 50; interchanging only the names of *possessor* and *expectant*, in their application to the lives, denoted by the letters A , and C .

QUESTION LIH.

The probability, that A and B , considered as possessors, shall both die before, either D or C ; and that D , considered as the expectant, shall die before C , the survivor; also the present worth of C 's interest, in an estate, or legacy, worth \mathfrak{P} pounds; dependent on this order of survivorship; are required.

SOLUTION.

Here the probabilities, that D shall be survived, by

$$C, \text{ is } 1 - \frac{v}{2n}$$

$$A, \text{ \& } C, \text{ is } 1 - \frac{v}{2n} - \frac{v}{2f} + \frac{v^2}{3nf}$$

$$B \text{ \& } C, \text{ is } 1 - \frac{v}{2n} - \frac{v}{2m} + \frac{v^2}{3nm}$$

$$A, B \text{ \& } C, \text{ is } 1 - \frac{v}{2n} - \frac{v}{2m} - \frac{v}{2f} + \frac{v^2}{3nf} + \frac{v^2}{3mf} + \frac{v^2}{3nf} - \frac{v^3}{4fmm}$$

by quest.

22;	31;	33;
-----	-----	-----

which

which present worths, are to be subtracted, and added, in the same manner, as the beforegoing probabilities;

$$\text{Now } vP = \frac{vw}{6rn} \times \frac{1}{2t} + \frac{1}{2m} =$$

$$vP = \frac{vw}{6rn} \times \frac{m+t \times 2}{4mt};$$

$$\text{And } vP = \frac{vw}{6rn} \times \frac{m+t \times 2}{4mt} - v =$$

$$vP = \frac{vw}{6rn} \times \frac{m+t \times 2}{4mt} - \frac{v}{4mt};$$

$$\text{Their difference is, therefore, } vP = \frac{vw}{6rn} \times \frac{v}{4mt};$$

Whence, the result will be

$$\{ vP = \frac{vw}{6rn} \times v + \frac{vw}{6rn} \times v - v \times 2 \times \frac{1}{4mt};$$

$$\text{or } vP + \frac{2vw}{6rn} = \frac{3v^2}{6rn} \times \frac{1}{4mt}; \text{ that is,}$$

$$\{ vP + \frac{2n-3v}{6rn} \times vw \times \frac{1}{4mt}$$

$$\text{or } vP + \frac{2n-3v}{6rn} \times v \times \frac{v}{4mt}; \text{ and therefore,}$$

$$vP + \frac{2n-3v}{6rn} \times v \times \frac{v}{4mt}, \text{ will be the present worth required.}$$

EXAMPLE.

If *A* (aged 43) and *B* (aged 54) are possessed of an estate of 1*l.* per annũm; which is to descend to *D* (aged 76) if he survives them both; and will, after that, come into the possession of *C* (aged 66) if he be alive at the death

death of D ; required the probability that he (C) has of becoming possessed of the estate; and the present worth of his interest therein.

Here $vD = 6,215$; $n = 20$; $v = 10$; $r = 1,04$; $P = 25$; $m = 32$; $t = 43$.

Then $\frac{1}{3} = 0,3333$; $\frac{10}{4,20} = 0,125$; $0,3333 \times 0,125 = 0,20833$ and $\left(\frac{0,20833 \times 10 \times 10}{32 \times 43} = \right)$

$0,01514$, the probability of C 's becoming possessed of the estate: also $2 \times 20 - 3 \times 10 = 10$; $\frac{10 \times 10}{6 \times 1,04 \times 20} = 0,801$;

and $6,215 + 0,801 = 7,016$;

Then $\left(\frac{7,016 \times 10 \times 25}{4 \times 32 \times 43} = \right) 0,31875$ will be the present worth of C 's interest, in the estate.

COROL. I.

If A and B are of equal ages; then $m = t$; and the answers will be $\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{mm}$; and

$$vD + \frac{2n - v}{6rn} \times v \times \frac{vD}{4mm}$$

COROL. II.

If B and C are of equal ages; then $m = n$; and the answers will become $\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{nt}$, and

$$vD + \frac{2n - v}{6rn} \times v \times \frac{vD}{4nt}$$

COROL.

COROL. III.

If C , and D , are of equal ages; then $v\mathfrak{P} = \mathfrak{D}$, and $v = v$; whence the answers will become

$$\left(\frac{1}{3} - \frac{1}{4} \times \frac{vv}{mt} \text{ or } \frac{vv}{12mt}\right), \text{ and } \mathfrak{D} = \frac{v}{6r} \times \frac{v\mathfrak{P}}{4mt}$$

COROL. IV.

[If A , B , and C , are of equal ages; then $t = m = n$; and the answers will be $\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{ns}$, and

$$v\mathfrak{P} + \frac{2n - 3v}{6rn} \times v \times \frac{v\mathfrak{P}}{4nn}.$$

COROL. V.

If B , C , and D , are of equal ages; then $v\mathfrak{P} = \mathfrak{D}$, and $m = n = v$; whence the answers will be

$$\left(\frac{1}{3} - \frac{1}{4} \times \frac{v}{t} \text{ or } \frac{v}{12t}\right), \text{ and } \mathfrak{D} = \frac{v}{6r} \times \frac{\mathfrak{D}}{4t}$$

QUESTION LIV.

The probability, that A and D , considered as possessors, shall both die before, either B , or C ; and that B , considered as expectant, shall die before C , the survivor; also the present worth of C 's interest, in an estate, or legacy, worth \mathfrak{P} pounds, dependent upon this order of survivorship; are required.

SOLU-

SOLUTION.

Here the probability, that *B* shall be survived, by

$$C, \text{ is } \frac{\pi}{2m}$$

$$A, \& C, \text{ is } \frac{\pi}{2m} - \frac{\pi^2}{6nm}$$

$$D, \& C, \text{ is } \frac{v}{2m} - \frac{v^2}{6nm}$$

$$A, D, \& C, \text{ is } \frac{v}{2m} - \frac{v^2}{6nm} - \frac{v^2}{6nm} + \frac{v^3}{12nm}$$

$$\left. \begin{array}{l} 21: \\ 31: \\ 31: \\ 33: \end{array} \right\} \text{by question}$$

which

COROL. IV.

If A , B , and C , are of equal ages; then $t = m = n$;
and the answers will be $nn - vv + \frac{v^3}{2n} \times \frac{1}{6nn}$, and

$$P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2nn}.$$

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$
and $P = P$; whence the answers will be, $\frac{v}{12t}$ and

$$P - \frac{v}{6r} \times \frac{P}{4t}.$$

QUESTION LV.

The probability, that B and D , considered as possessors, shall both die before either A or C ; and that A , considered as expectant, shall die before C , the survivor; also the present worth of C 's interest, in an estate or legacy, worth P pounds, dependent upon this order of survivorship; are required.

SOLU-

SOLUTION.

Here the probability that *A* shall be survived, by

$$\begin{array}{rcl}
 C, \text{ is } \frac{n}{2t} & - & \frac{nn}{6mt} \\
 B, \& C, \text{ is } \frac{n}{2t} & - & \frac{nn}{6mt} \\
 D, \& C, \text{ is } \frac{v}{2t} & - & \frac{vv}{6mt} \\
 B, D, \& C, \text{ is } \frac{v}{2t} & - & \frac{vv}{6mt} + \frac{v^3}{12ntm}
 \end{array}$$

by quest.

$$\begin{array}{rcl}
 21; & 31; & 31; \\
 33; & 31; & 31;
 \end{array}$$

Which probabilities, being subtracted and added, in the manner above directed, will give

$$\left(\frac{nn}{6tm} - \frac{vv}{6tm} + \frac{v^3}{12ntm} \text{ or } \frac{nn-vv+\frac{v^3}{2n}}{6tm} \times \frac{1}{6tm} \right)$$

for the probability required; and consequently

$$\left(P - \frac{n}{6r} \times \frac{nP}{2tm} - P - \frac{v}{6r} \times \frac{vP}{2tm} \right. \\ \left. + P - \frac{v}{6r} \times \frac{vvP}{4mtn} \right)$$

$$\text{or) } P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2mt}$$

will be the present worth of C's interest in that estate.

COROL.

Since the result, above found, is exactly the same, with that of the last question; therefore, the numeral process, and the answers to all the varieties, occasioned by supposing two or three of the given lives to be of equal ages, will be the same, as in the example and corollaries, to that question.

QUESTION LVI.

The probability that A and C, considered as possessors, shall both die before either D or B; and that D, considered as expectant, shall die before B, the survivor; also the present worth of B's interest, in an estate or legacy, worth P pounds, dependent upon this order of survivorship; are required,

SOLU.

SOLUTION.

Here the probability, that *D* shall be survived, by

$$B, \text{ is } 1 - \frac{v}{2n}$$

$$A, \text{ \& } B, \text{ is } 1 - \frac{v}{2m} - \frac{v}{2t} + \frac{vv}{3mt}$$

$$C, \text{ \& } B, \text{ is } 1 - \frac{v}{2n} - \frac{v}{2m} + \frac{vv}{3nm}$$

$$A, C, \text{ \& } B, \text{ is } 1 - \frac{v}{2n} - \frac{v}{2m} - \frac{v}{2t} + \frac{vv}{3nt} + \frac{vv}{3nm} - \frac{vv}{3mt} - \frac{v^3}{4nm}$$

$$\left. \begin{array}{l} 223 \\ 313 \\ 313 \\ 333 \end{array} \right\} \text{by quest.}$$

Which probabilities, being subtracted, and added, according to the directions above-mentioned, will give $\left(\frac{vv}{3nt} - \frac{v^3}{4nm} = \right) \frac{1}{3} - \frac{v}{3n} \times \frac{vv}{nt}$, for the probability required.

Now

Now the present worths, corresponding to the above probabilities are

$$\left. \begin{array}{l} 23: \\ 32: \\ 32: \\ 34: \end{array} \right\} \text{by schol. to question}$$

which

which present worths are to be subtracted and added, in the same manner, as the before going probabilities; and the result will be

$$\left(\frac{vv}{6rn} \times \frac{1}{2t} - \frac{vv}{6rn} \times 2m - v - \frac{vv}{6rn} \times n - u \times 2 \times \frac{1}{4mt} \right)$$

Or)

$$\left\{ \frac{vv}{6rn} \times 2m - v - \frac{vv}{6rn} \times 2m - v + \frac{vv}{6rn} \times n - u \times 2 \right\} \times \frac{1}{4mt}$$

And therefore,

$$\left\{ \frac{vv}{6rn} \times 2m - v - \frac{vv}{6rn} \times 2m - v + \frac{vv}{6rn} \times n - u \times 2 \right\} \times \frac{1}{4mt}$$

will be the present worth required.

EXAMPLE.

C (aged 66) is possessed of an estate of 1£. per annum, in which his wife A aged 43 is jointured; after their decease, it will devolve to D (aged 76) the uncle of C, if he be then alive; now D has a daughter B (aged 54) who will succeed to the same after D's decease, if she be then alive; required the probability that B has of becoming possessed of the estate, and the present worth of her interest therein.

174 MATHEMATICAL

Here $vM = 6,925$; $vP = 6,215$; $v = 10$; $n = 20$; $m = 32$; $t = 43$; $r = 1,04$; $P = 25$.

Then $\frac{1}{3} = 0,3333$; $\frac{10}{4.32} = 0,0781$;

And $0,3333 - 0,0781 = 0,2552$; therefore

$\left(\frac{0,2552 \times 10 \times 10}{20 \times 43} \right) = 0,02967$ will be the probabi-

lity required.

Also $\frac{10 \cdot 10}{6,1,04 \cdot 20} = 0,801$; $\frac{10 \cdot 10}{6,1,04 \cdot 32} = 0,501$; $6,925$

$- 0,501 = 6,424$; $2m - v = 54$; $6,215 - 0,801 = 5,414$; $6,424 \times 2 \times 32 = 411,136$; $5,414 \times 54 = 292,356$; $0,801 \times 20 - 10 \times 2 = 16,02$; and $411,136 - 292,356 + 16,02 = 134,8$;

Therefore $\frac{134,8 \times 25}{4 \times 32 \times 43} = 0,6126$ will be the present worth required.

COROL. I.

If A , and B , are of equal ages; then $t = m$; and the

answers will be $\frac{1}{3} - \frac{v}{4m} \times \frac{vv}{nm}$ and

$$\left. \begin{aligned} & vM - \frac{vv}{6rm} \times 2m - vP - \frac{vv}{6rn} \times 2m - v \\ & + \frac{vv}{6ru} \times n - v \times 2 \end{aligned} \right\} \times \frac{P}{4nm}.$$

COROL. II.

If B and C are of equal ages; then $m = n$; and $vM =$

vP whence the answers will become $\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{n}$ and

$$\text{and } (vD - \frac{vv}{6rn} \times v + \frac{vv}{6rn} \times n - v \times 2 \times \frac{D}{4nt}, \text{ or})$$

$$vD - \frac{vv}{6rn} + \frac{2v}{6rn} \times n - v \times \frac{vD}{4nt};$$

$$\text{That is } (vD - \frac{vv}{6rn} + \frac{2nv - 2vv}{6rn} \times \frac{vD}{4nt}, \text{ or})$$

$$vD + \frac{2n - 3v}{6rn} \times v \times \frac{vD}{4nt}.$$

COROL. III.

If C and D, are of equal ages; then $n = v$, and $vD = D$; whence the answers will become

$$\frac{1}{3} - \frac{v}{4n} \times \frac{v}{t}, \text{ and}$$

$$(vD - \frac{vv}{6rm} \times 2m - D - \frac{v}{6r} \times 2m - v \times \frac{D}{4nt},$$

$$\text{or) } vD - \frac{vv}{6rm} - D - \frac{v}{6r} \times \frac{2m - v}{2m} \times \frac{D}{2t}.$$

COROL. IV.

If A, B, and C, are of equal ages; then $t = m = n$, and $vD = vD$; whence the answers will become,

$$\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{nn}, \text{ and}$$

$$(vD - \frac{vv}{6rn} + \frac{2v}{6rn} \times n - v \times \frac{vD}{4nt},$$

$$\text{or) } vD + \frac{2n - 3v}{6rn} \times v \times \frac{vD}{4nt}.$$

COROL. V.

If B , C , and D , are of equal ages; then $m=n=v$, and $\frac{v}{3} = \frac{v}{3} = \frac{v}{3}$; whence the answers will be-

come $\left(\frac{1}{3} - \frac{v}{4v} \times \frac{v}{t} \text{ or } \right) \frac{v}{12t}$, and

$$\left(\frac{v}{6r} \times 2v - \frac{v}{6r} \times 2v - v \times \frac{v}{4vt} \right), \text{ or}$$

$$\frac{v}{6r} \times \frac{v}{4t}.$$

QUESTION LVII.

The probability, that A and D , considered as possessors, shall both die before either C or B ; and that C , considered as expectant, shall die before B , the survivor; also the present worth of B 's interest, in an estate or legacy, worth \mathcal{P} pounds, dependent upon this order of survivorship; are required.

SOLU.

for the probability of the event

SOLUTION.

Here the probability that C shall be survived, by

$$B, \text{ is } 1 - \frac{n}{2m}$$

$$A, \text{ \& } B, \text{ is } 1 - \frac{n}{2m} - \frac{n}{2l} + \frac{n^2}{3ms}$$

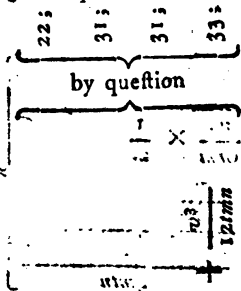
D, \& B, is

A, D, \& B, is

Which probabilities, being subtracted and added, according to the abovementioned di-

rections, will give $1 - \frac{n}{2l} - \frac{nv}{6m} + \frac{v^3}{12mn}$ or $1 - \frac{n}{3m} + \frac{nv}{6m} - \frac{v^3}{12mn}$ for the probability required.

Now



$$1 - \frac{n}{2l} - \frac{nv}{6m} + \frac{v^3}{12mn}$$

$$nD - \frac{nn}{6rm} - D - \frac{v}{6r} \times \frac{2m-v \times v}{2mn} \times \frac{1}{2t};$$

therefore the required present worth will be,

$$nD - \frac{nn}{6rm} - D - \frac{v}{6r} \times \frac{2m-v \times v}{2mn} \times \frac{1}{2t}.$$

EXAMPLE.

There are two brothers *D* (aged 76) and *C* (aged 66) *D*, the elder brother, is possessed of a paternal estate of 1 £. per annum, and hath a daughter *A* (aged 43) who is to possess the same, during her life, if she survives her father; but, after their decease, if *C*, the younger brother, be then alive, he will become possessed thereof, and may leave it to his wife *B* (aged 54), the probability of *B*'s becoming possessed of the estate, together with the present worth of her interest therein, are required.

Here $nD - \frac{nn}{6rm} = 7,888$; $D - \frac{v}{6r} = 2,715$;
 $v=10$; $n=20$; $m=32$; $t=43$; $2m-v=54$; and
 $2mn=1280$;

Then $\frac{20}{3.32} = 0,2083$; and $0,5 - 0,2083 = 0,2916$;
 also $0,2916 \times 20 = 5,833$;

Again $\frac{10}{2.32} = 0,15625$; and $1 - 0,15625 =$
 $0,84375$; also $\frac{0,84375 \times 10 \times 10}{6.20} = 0,703$;

Whence $(5,833 - 0,703 \text{ or } 5,130 \times \frac{1}{43} = 0,1193$,
 will be the probability required.

Now $\frac{2,715 \times 54 \times 10}{1280} = 1,145$; and $7,888 - 1,145$
 $= 6,743$;

Whence $\left(\frac{6,743 \times 25}{2 \times 43} \right) 1,960$ will be the present
 worth required.

COROL. I.

If A , and B , are of equal ages; then $t=m$; and the answers will become:

$$\frac{1}{2} - \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n} \times \frac{1}{m}, \text{ and}$$

$$P - \frac{nn}{6rm} - P - \frac{v}{6r} \times \frac{2m-v \times v}{2mn} \times \frac{P}{2m}.$$

COROL. II.

If B , and C , are of equal ages; then $m=n$, and $n=t$:

whence $(\frac{1}{2} - \frac{1}{3} \times n - 1 - \frac{v}{2n} \times \frac{vv}{6n} \times \frac{1}{t})$ or

$$\frac{n}{6} - 1 - \frac{v}{2n} \times \frac{vv}{6n} \times \frac{1}{t},$$

That is $n - 1 - \frac{v}{2n} \times \frac{vv}{n} \times \frac{1}{6t}$ will be the probability; and,

$P - \frac{n}{6r} - P - \frac{v}{6r} \times \frac{2n-v \times v}{2nn} \times \frac{P}{2t}$, will be the present worth required.

COROL. III.

If C , and D , are of equal ages; then $n=v$; and

$$(\frac{1}{2} - \frac{v}{3m} \times v - 1 - \frac{v}{2m} \times \frac{v}{6} \times \frac{1}{t}), \text{ or}$$

$$3 - \frac{2v}{m} - 1 - \frac{v}{2m} \times \frac{v}{6t}, \text{ that is}$$

$(2 - \frac{3v}{2m} \times \frac{v}{6t} \text{ or } \frac{1}{3} - \frac{v}{4m} \times \frac{v}{t})$, will be the probability; and, $\frac{1}{6} - \frac{v}{6rm} - \frac{v}{6r} \times \frac{2m-v}{2m} \times \frac{1}{2t}$, will be the present worth required.

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$, and $\frac{1}{6} = \frac{1}{6}$; whence

$(\frac{1}{6} - 1 - \frac{v}{2n} \times \frac{vv}{n} \times \frac{1}{6n} \text{ or } \frac{1}{6} - 1 - \frac{v}{2n} \times \frac{vv}{6nn})$, will be the probability; and $\frac{1}{6} - \frac{n}{6r} - \frac{v}{6r} \times \frac{2n-v \times v}{2nn} \times \frac{1}{2n}$, will be the present worth required.

COROL. V.

If B , C , and D , are of equal ages; the $m=n=v$, and $\frac{1}{6} = \frac{1}{6}$; whence $(2 - \frac{1}{2} \times \frac{v}{6t} \text{ or } \frac{v}{12t})$, will be the probability, and

$(\frac{1}{6} - \frac{v}{6r} - \frac{v}{6r} \times \frac{2v-v}{2v} \times \frac{1}{2t} =)$
 $\frac{1}{6} - \frac{v}{6r} \times \frac{1}{4t}$, will be the present worth required.

QUESTION

QUESTION LVIII.

The probability, that *C* and *D*, considered as possessors, shall both die before either *A* or *B*, and that *A*, considered as expectant, shall die before *B*, the survivor; also the present worth of an estate or legacy, worth *P*, pounds, dependent upon this order of survivorship; are required.

SOLUTION.

Here the probability, that *A* shall be survived, by

$$B, \text{ is } \frac{m}{2t}$$

$$C, \text{ \& } B, \text{ is } \frac{n}{2t} - \frac{mn}{6mt}$$

$$D, \text{ \& } B, \text{ is } \frac{v}{2t} - \frac{uv}{6mt}$$

$$C, D, \text{ \& } B, \text{ is } \frac{v}{2t} - \frac{uv}{6mt} + \frac{uv^2}{12mnt}$$

$$\underbrace{\begin{matrix} 211 \\ 311 \\ 312 \\ 322 \end{matrix}}_{\text{by question}}$$

Which probabilities, being subtracted and added, according to the directions before mentioned, will give $\left(\frac{m}{2t} - \frac{n}{2t} + \frac{uv}{6mt} - \frac{uv^2}{12mnt} \right) +$

$$\frac{v^3}{12mnt} \text{ or } \frac{m}{2t} - 1 - \frac{n}{3m} \times \frac{v}{2t} - 1 - \frac{v}{2m} \times \frac{uv}{6mt}$$

That

$$\text{That is, } m-1 = \frac{n}{3m} \times n-1 = \frac{v}{2m} \times \frac{vv}{3n} \times \frac{1}{2t}$$

for the probability required.

And the present worths corresponding to these proba-

$$\text{bilities will be } \frac{A}{t} - \frac{A}{t} + \frac{A}{t} - \frac{n}{6r} \times \frac{n}{2mt} -$$

$$\frac{A}{t} - \frac{v}{6r} \times \frac{v}{2nt} + \frac{A}{t} - \frac{v}{6r} \times \frac{vv}{4nmt}$$

$$\text{But, } \frac{A}{t} - \frac{v}{6r} \times \frac{v}{2nt} = \frac{A}{t} - \frac{v}{6r} \times \frac{v}{2nt} \times \frac{v}{2m}$$

$$= \frac{A}{t} - \frac{v}{6r} \times 1 - \frac{v}{2m} \times \frac{v}{2nt};$$

Whence the above expression will become

$$\left(\frac{A}{t} - \frac{A}{t} + \frac{A}{t} - \frac{n}{6r} \times \frac{n}{2mt} - \right.$$

$$\left. \frac{A}{t} - \frac{v}{6r} \times 1 - \frac{v}{2m} \times \frac{v}{2nt}, \text{ or:} \right\}$$

$$\frac{A}{t} - \frac{A}{t} + \left. \frac{A}{t} - \frac{n}{6r} \times n - \frac{v}{6r} \times \frac{2m-v}{2n} \times v \times \frac{1}{2m} \right\} \times \frac{n}{t}$$

therefore the present worth of B's interest in the estate will be,

$$\frac{A}{t} - \frac{A}{t} + \left. \frac{A}{t} - \frac{n}{6r} \times n - \frac{v}{6r} \times \frac{2m-v}{2n} \times v \times \frac{1}{2m} \right\} \times \frac{n}{t}$$

EXAMPLE

EXAMPLE.

D (aged 76) is possessed of an estate of 1*£*. per annum; which (on his decease) will descend to his brother C (aged 66); and, after both their deaths, to A , the son of C (aged 43) if he be then alive; who may leave it to B , his wife, aged 54; the probability of her coming into possession of, and the present value of her interest in, the estate, are required.

Here $w=10$; $n=20$; $m=32$; $t=43$; $M=10,757$;

$$N=7,673; P-\frac{n}{6r}=4,468; Q-\frac{v}{6r}=2,715;$$

and $R=25$.

$$\text{Then } \frac{20}{3 \times 32} = 0,2083; 1-0,2083=0,7917; 0,7917 \times 20 = 15,834;$$

$$\text{And } \frac{10}{2 \times 32} = 0,15625; 1-0,15625 = 0,84375;$$

$$0,84375 \times \frac{10 \times 10}{3 \times 20} = 1,406;$$

Therefore $(32 - 15,834 - 1,406 \text{ or } 14,76 \times \frac{1}{2 \times 43}) = 0,172$, will be the probability required.

$$\text{Again } 4,468 \times 20 = 89,36; \frac{2m-v}{2n} = \left(\frac{54}{40} = \frac{27}{20} = \right) 1,35; 2,715 \times 1,35 \times 10 = 36,6525; 89,36 - 36,6525 = 52,7075; \text{ and } \frac{52,7075}{2 \cdot 32} = 0,824.$$

Then $10,757 - 7,673 + 0,824 = 3,908$: therefore $\left(\frac{3,908 \times 25}{43} = \right) 2,272$ will be the present worth required.

COROL.

COROL. I.

If A , and B , are of equal ages; then $t=m$; and the answers will become,

$$m-1 = \frac{n}{3m} \times n-1 = \frac{v}{2m} \times \frac{vv}{3n} \times \frac{1}{2m}, \text{ and}$$

$$\left. \begin{aligned} P-1 &= P-1 = \frac{n}{6r} \times \frac{n}{2m} \\ P-1 &= \frac{v}{6r} \times \frac{2m-v}{2n} \times v \times \frac{1}{2m} \end{aligned} \right\} \times \frac{P}{m}$$

COROL. II.

If B , and C , are of equal ages; then $m=n$, and $P=P$;

Whence $(n-1 = \frac{1}{3} \times n-1 = \frac{v}{2n} \times \frac{vv}{3n} \times \frac{1}{2n}, \text{ or})$

$$\frac{n}{3} - 1 = \frac{v}{2n} \times \frac{vv}{3n} \times \frac{1}{2n}; \text{ that is}$$

$n-1 = \frac{v}{2n} \times \frac{vv}{n} \times \frac{1}{6t}$, will be the probability; and

$$(P-1 = \frac{n}{6r} \times n-1 = \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2nt},$$

Or) $P-1 = \frac{n}{6r} - \frac{v}{6r} \times \frac{2n-v \times v}{2nn} \times \frac{P}{2t}$, will be the present worth required.

COROL.

COROL. III.

If C , and D , are of equal ages; then $x = v$, and $P = D$; whence

$$(m-1) - \frac{v}{3m} \times v - 1 - \frac{v}{2m} \times \frac{v}{3} \times \frac{1}{2r}, \text{ or}$$

$$m - 4 - \frac{3v}{2m} \times \frac{v}{3} \times \frac{1}{2r}, \text{ will be the probability; and}$$

$$D - D + U - \frac{v}{6r} \times \frac{v}{2m} - \left. \begin{array}{l} \\ \\ \end{array} \right\} \times \frac{P}{r}$$

$$U - \frac{v}{6r} \times v \times \frac{2m-v}{2v} \times \frac{1}{2m}$$

or) $D - U + U - \frac{v}{6r} \times \frac{3v-2m}{4m} \times \frac{P}{r}$ will be the present worth required.

COROL. IV.

If A , B , and C , are of equal ages; then $r = m = v$, and

$$D = D; \text{ whence } (n-1) - \frac{v}{2n} \times \frac{v}{n} \times \frac{1}{6n}, \text{ or } \frac{1}{6}$$

$$- 1 - \frac{v}{2n} \times \frac{vv}{6nn}, \text{ will be the probability; and}$$

$D - \frac{v}{6r} - D - \frac{v}{6r} \times \frac{2n-v}{2nn} \times \frac{P}{2n}$ will be the present worth required.

COROL.

Now the present worths, corresponding to the above probabilities are

$$\begin{array}{c}
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t}
 \end{array}
 \quad
 \begin{array}{c}
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \\
 \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t}
 \end{array}$$

Which present worths, being subtracted and added, in the same manner, as the before-going probabilities; the result will be $\left(\frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t} \right) + \frac{v^2}{v} - \frac{v^2}{v} \times \frac{1}{2t}$

by schol. to question
 23: 32: 32: 34:

COROL. II.

If B , and C , are of equal ages; then $n = v$, and $vB = vC$; whence the answers will become $\frac{1}{3} - \frac{v}{4t} \times \frac{vv}{n}$, and

$$2 \times \frac{vB - vC}{v} + \frac{vB - \frac{vv}{6n}}{6n} \times 2n + v - 2t + \frac{vv}{6n} \times n - v \times 2 \times \frac{1}{4nt}.$$

COROL. III.

If C , and D , are of equal ages; then $n = v$, and $vC = vD$; whence the answers will become $\frac{1}{3} - \frac{v}{4t} \times \frac{v}{n}$, and

$$2 \times \frac{vC - vD}{v} + \frac{vC - \frac{vv}{6n}}{6n} \times 2n - \frac{v}{6n} \times 2t - v \times \frac{1}{4nt}.$$

COROL.

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$, and $vP = vB = vD$; whence the answers will be-

come $\frac{1}{3} - \frac{v}{4n} \times \frac{vv}{nn}$, and

$$\left(vD - \frac{vv}{6rn} \times v + \frac{vv}{6rn} \times n - v \times 2 \times \frac{P}{4nn} \right), \text{ or}$$

$$vD + \frac{2n - 3v}{6rn} \times v \times \frac{vP}{4nn}.$$

COROL. V.

If B , C , and D , are of equal ages; then $m=n=v$, and $vB = vC = vD$; whence the answers will become

$\frac{1}{3} - \frac{v}{4t}$, and

$$\left(P \times \frac{vP - D}{v} + D - \frac{v}{6r} \times 3v - 2t \times \frac{1}{4vt} \right), \text{ or}$$

$$vP - D + D - \frac{v}{6r} \times \frac{3v - 2t}{4t} \times \frac{P}{v}.$$

QUESTION LX.

The probability, that B and D , considered as possessors, shall both die before either C or A ; and that C , considered as expectant, shall die before A , the survivor; also the present worth of an estate, or legacy, worth P pounds, dependent upon this order of survivorship; are required.

[SOLUTION.]

Here the probability that C shall be unriveted, by

$$A, \text{ is } 1 - \frac{n}{2t}$$

$$B, \text{ \& } A, \text{ is } 1 - \frac{n}{2m} - \frac{n}{3mt}$$

$$D, \text{ \& } A, \text{ is}$$

$$B, D, \text{ \& } A, \text{ is}$$

Which probabilities, (being subtracted and added, as above directed) will give

$$\left(\frac{n}{2m} - \frac{n}{3mt} - \frac{v}{6mn} + \frac{v^3}{12m^3n} \right) \frac{1}{2} - \frac{v}{3t} \times \frac{v}{6mn} - \frac{v}{6mn} + \frac{v^3}{12m^3n}$$

Now $\frac{1}{2} - \frac{n}{3t} \times \frac{v}{6mn} - \frac{v}{6mn} + \frac{v^3}{12m^3n}$ for the probability required.

That is,

$$\frac{1}{2} - \frac{n}{3t} \times \frac{v}{6mn} - \frac{v}{6mn} + \frac{v^3}{12m^3n}$$

by question

Now the present worths corresponding to the above probabilities are

$$\left. \begin{aligned}
 & \frac{^nP}{n} \\
 & \frac{^nM}{n} - \frac{^nM}{n} - \frac{nn}{6rm} \times \frac{1}{2t} \\
 & \frac{^nA}{n} - \frac{^nA}{n} - \frac{v}{6r} \times \frac{v}{2tn} \\
 & \frac{^nB}{n} - \frac{^nB}{n} - \frac{v}{6r} \times \frac{m+t \times 2 - v \times v}{4mtn}
 \end{aligned} \right\} \begin{array}{l} \text{by question} \\ 235 \\ 325 \\ 325 \\ 345 \end{array}$$

$$\left. \begin{aligned}
 & \text{Now } \frac{^nB}{n} - \frac{v}{6r} - \\
 & \frac{^nB}{n} - \frac{v}{6r} \times \frac{m+t \times 2 - v}{2m} \\
 & \left(\frac{^nB}{n} - \frac{v}{6r} \times 1 - \frac{2m+t \times 2 - v}{2m} \times \frac{v}{2tn} \text{ or} \right) \\
 & - \frac{^nB}{n} - \frac{v}{6r} \times \frac{2t-v}{2m} \times \frac{v}{2tn}
 \end{aligned} \right\} \times \frac{v}{2tn} \text{ will be equal to}$$

Therefore, if the above present worths be subtracted and added (as usual) the result will become,

$$\left(\frac{^nP}{n} - \frac{^nM}{n} + \frac{^nM}{n} - \frac{nn}{6rm} \times \frac{1}{2t} \right.$$

$$\left. - \frac{^nA}{n} - \frac{v}{6r} \times \frac{2t-v}{2m} \times \frac{v}{2tn}, \text{ or} \right)$$

$$\frac{^nP}{n} - \frac{^nM}{n} +$$

$$\frac{^nM}{n} - \frac{nn}{6rm} - \frac{^nA}{n} - \frac{v}{6r} \times \frac{2t-v \times v}{2mn} \times \frac{1}{2t}$$

196 MATHEMATICAL

Therefore the present worth of an estate, worth \mathcal{P} pounds, will be

$$\mathcal{P} \times \left\{ \frac{\mathcal{P} - \mathcal{P}}{\mathcal{P}} + \frac{\mathcal{P} - \frac{\mathcal{P}}{Or} - \mathcal{P} - \frac{\mathcal{P}}{Or} \times \frac{2t - v \times v}{2mn} \times \frac{1}{2t} \right\}$$

EXAMPLE.

D (aged 76) and his daughter B (aged 54) are in possession of an estate (worth \mathcal{P} pounds) which, after both are dead, will devolve to C , the brother of D (aged 66) if he be then living; D has a wife A (aged 43) who will, if she survives him, inherit after her husband; what is the present worth of A 's interest in the estate?

Here $\mathcal{P} = 10,838$; $\mathcal{P} = 9,891$; $n = 20$;

$$\mathcal{P} - \frac{\mathcal{P}}{Or} = 7,888; \mathcal{P} - \frac{\mathcal{P}}{Or} = 2,715; t = 43;$$

$$v = 10; m = 32; \frac{2t - v \times v}{2mn} = \left(\frac{76 \times 10}{2 \cdot 32 \cdot 20} \right) \frac{19}{32}$$

$$\text{Then } 10,838 - 9,891 = 0,947; \frac{0,947}{20} = 0,04735;$$

$$2,715 \times \frac{19}{32} = 1,612; 7888 - 1,612 = 6,276; \frac{6,276}{86}$$

$= 0,07298$; $0,04735 + 0,07298 = 0,12033$; and $(0,12033 \times 25 =) 3,00825$, will be the present worth required.

If the probability, of A 's becoming possessed of the estate, had been required; then $\frac{n}{3t} = \left(\frac{20}{3 \cdot 43} \right)$

$$\frac{20}{129}; \frac{v}{2t} = \frac{10}{86}; \frac{1}{2} - \frac{20}{129} = \frac{89}{258}; 1 - \frac{10}{86} =$$

$$\frac{76}{86}; \text{ and } \frac{vv}{6n} = \left(\frac{10 \cdot 10}{6 \cdot 20} \right) \frac{5}{6}$$

Therefore

Therefore $\frac{89}{258} \times 20 = 6,899$; $\frac{76}{86} \times \frac{5}{6} = 0,736$;
 $6,899 - 0,736 = 6,163$;

And $\left(\frac{6,163}{32} =\right) 0,193$ will be the probability required.

COROL. I.

If A , and B , are of equal ages; then $nA = nB$, and $t = m$; whence the answers will become

$$\frac{1}{2} - \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n} \times \frac{1}{m}, \text{ and}$$

$$nB - \frac{nn}{6rm} - B - \frac{v}{6r} \times \frac{2m-v \times v}{2mn} \times \frac{B}{2m}.$$

COROL. II.

If B , and C , are of equal ages; then $nB = nC$, and $m = n$; whence the answers will become

$$\left(\frac{1}{2} - \frac{n}{3t} \times n - 1 - \frac{v}{2t} \times \frac{vv}{6n} \times \frac{1}{n}, \text{ or,}\right)$$

$$\frac{1}{2} - \frac{n}{3t} - 1 - \frac{v}{2t} \times \frac{vv}{6nn}, \text{ and}$$

$$B \times \left\{ \frac{nB - B}{n} + \right.$$

$$B - \frac{n}{6r} - B - \frac{v}{6r} \times \frac{2t-v \times v}{2nn} \times \frac{1}{2t}.$$

COROL. III.

If C , and D , are of equal ages; then $x = v$; and the answers will become

$$\left(\frac{1}{2} - \frac{v}{3t} \times v - 1 - \frac{v}{2t} \times \frac{v}{6} \times \frac{1}{m}, \text{ or} \right)$$

$$2 - \frac{3v}{2t} \times \frac{v}{6m}, \text{ and}$$

$$P \times \left\{ \frac{vP - vP}{v} + \right.$$

$$\left. vP - \frac{vv}{6rm} - P - \frac{v}{6r} \times \frac{2t-v}{2m} \times \frac{1}{2t} \right\}$$

COROL. IV.

If A , B , and C , are of equal ages; then $t = m = x$, and $vP = vP = P$; whence the answers will become

$$\left(\frac{1}{2} - \frac{1}{3} \times x - 1 - \frac{v}{2n} \times \frac{vv}{6n} \times \frac{1}{n}, \text{ or} \right)$$

$$\frac{1}{6} - 1 - \frac{v}{2n} \times \frac{vv}{6n}, \text{ and}$$

$$P - \frac{n}{6r} - P - \frac{v}{6r} \times \frac{2n-v}{2nn} \times \frac{P}{2n}.$$

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$, and $vP = P = P$; whence the answers will become,

$$\frac{1}{3} - \frac{v}{4t}, \text{ and}$$

 $(P \times$

$$(P \times \left\{ \frac{v^2 - P}{v} + \frac{P - \frac{v}{6r} - P - \frac{v}{6r} \times \frac{2t-v}{2v} \times \frac{1}{2t}, \text{ or} \right\} \\ P \times \frac{v^2 - P}{v} + P - \frac{v}{6r} \times 1 - \frac{2t-v}{2v} \times \frac{1}{2t}$$

That is,

$$(P \times \frac{v^2 - P}{v} + P - \frac{v}{6r} + \frac{2v-2t}{2v} \times \frac{1}{2t}, \text{ or}) \\ \frac{v^2 - P}{v} + P - \frac{v}{6r} \times \frac{3v-2t}{4t} \times \frac{P}{v}$$

QUESTION LXI

The probability, that *C* and *D*, considered as possessors, shall both die before either *B* or *A*; and that *B*, considered as expectant, shall die before *A*, the survivor; also the present worth of an estate, or legacy, worth *P* pounds, dependent upon this order of survivorship; are required.

SOLUTION.

Hence, the probability, that B shall be survived, by

$$A, \text{ is, } 1 - \frac{m}{2f}$$

C & A, is,

$$\frac{m}{2m} - \frac{m}{6fm}$$

D & A, is,

$$\frac{v}{2m} - \frac{qv}{6fm}$$

C, D & A, is,

$$\frac{v}{2m} - \frac{qv}{6fm} - \frac{qv}{6fm} + \frac{v^3}{12fm^2}$$

$$\left. \begin{array}{l} 22; \\ 31; \\ 31; \\ 33; \end{array} \right\} \text{by quest.}$$

Which probabilities (being subtracted and added, as before) will give $(1 - \frac{m}{2f} - \frac{m}{2m}$

$$+ \frac{m}{6fm} - \frac{qv}{6fm} + \frac{v^3}{12fm^2}, \text{ or } 1 - \frac{m}{2f} - \frac{m}{2m} \times \frac{m}{2m} - 1 - \frac{v}{2f} \times \frac{qv}{6fm}, \text{ for the}$$

probability required.

Now

Now since the present worths

$$\frac{mP}{m}, \frac{P}{m}, P - \frac{n}{6r} \times \frac{n}{2tm}, Q - \frac{v}{6r} \times \frac{v}{2nm},$$

and $Q - \frac{v}{6r} \times \frac{vv}{4mn}$, severally, correspond to the

probabilities $1 - \frac{m}{2t}, \frac{n}{2m}, \frac{nn}{6tm}, \frac{vv}{6nm}$, and $\frac{v^3}{12ntm}$;

therefore those present worths, being connected by the

proper signs, viz. $\left(\frac{mP}{m} - \frac{P}{m} + P - \frac{n}{6r} \times \frac{n}{2tm} \right.$

$$\left. - Q - \frac{v}{6r} \times \frac{v}{2nm} + Q - \frac{v}{6r} \times \frac{vv}{4mn} \right,$$

$$\text{or) } \frac{mP}{m} - \frac{P}{m} + P - \frac{n}{6r} \times \frac{n}{2tm}$$

$$- Q - \frac{v}{6r} \times 1 - \frac{v}{2t} \times \frac{v}{2nm}, \text{ will be the present}$$

worth, corresponding to that probability.

And, because m is a constant divisor in every term thereof, the present worth of an estate, worth P pounds, will be,

$$\frac{mP}{m} - \frac{P}{m} + P - \frac{n}{6r} \times \frac{n}{2t} - \left\{ \frac{v}{6r} \times 1 - \frac{v}{2t} \times \frac{v}{2n} \right\} \times \frac{P}{m},$$

which was required.

EXAMPLE.

D (aged 76) has jointured his second wife C (aged 66) in an estate of 1*l.* per annum; which, after both their deaths, will descend to D 's son B (aged 54) if he be then alive; B has, by deed, settled the same on his wife A , aged 43; required the probability of her becoming possessed of the estate, and the present worth of her interest therein.

K 5

Here

Here $\overline{A} = 12,578$; $\overline{B} = 7,673$; $\overline{D} = \frac{n}{6r} \times \frac{n}{2t}$
 $= 1,039$ (by quest. 4); $\overline{D} = \frac{v}{6r} = 2,715$; $\frac{v}{2t} =$
 $\frac{1}{18}$; $1 - \frac{1}{18} = (\frac{17}{18} =) \frac{17}{18}$; $\frac{v}{2n} = (\frac{1}{18} =) \frac{1}{18}$; $\overline{D} =$
 25 ; and $m = 32$. Then $2,715 \times \frac{17}{18} \times \frac{1}{18} = 0,600$;
 $12,578 - 7,673 + 1,039 - 0,600 = 5,344$; and
 $(5,344 \times \frac{2}{32} =) 4,175$, will be the present value of A 's
 interest in the estate.

Also $\frac{m}{2t} = 0,372$; $\frac{n}{3t} = \frac{20}{129}$; $1 - \frac{20}{129} = \frac{109}{129}$;
 $\frac{n}{2m} = (\frac{20}{64} =) \frac{5}{16}$; $\frac{v}{2t} = (\frac{1}{18} =) \frac{1}{18}$; $1 - \frac{5}{18} = \frac{13}{18}$;
 and $\frac{vv}{6nm} = (\frac{10 \cdot 10}{6 \cdot 20 \cdot 32} =) \frac{5}{192}$;

Then $\frac{109}{129} \times \frac{13}{18} = 0,264$; and $\frac{17}{18} \times \frac{1}{18} = 0,023$;

Therefore $(1 - 0,372 - 0,264 - 0,023 =) 0,341$,
 will be the probability of A 's becoming possessed of the
 estate.

COROL. I.

If A and B , are of equal ages; then $\overline{A} = \overline{B}$, and
 $t = m$; whence the answers will become

$$\frac{1}{2} - 1 - \frac{n}{3m} \times \frac{n}{2m} - 1 - \frac{v}{2m} \times \frac{v}{6nm},$$

And $\overline{D} = \overline{D} +$

$$\left. \begin{aligned} &\overline{D} - \frac{n}{6r} \times \frac{n}{2m} - \overline{D} - \frac{v}{6r} \times 1 - \frac{v}{2m} \times \frac{v}{2n} \end{aligned} \right\} \times \frac{\overline{D}}{m}.$$

COROL.

COROL. II.

If B, and C, are of equal ages; then $m = n$; and the answer will become $(1 - \frac{n}{2t} - 1 - \frac{n}{3t} \times \frac{1}{2} - 1 - \frac{v}{2t} \times \frac{vv}{6mn})$ or $1 - \frac{n}{3t} - 1 - \frac{v}{2t} \times \frac{vv}{6mn}$.

And $\frac{m}{2t} - \frac{1}{2} + \left\{ \frac{m}{6r} \times \frac{n}{2t} - \frac{1}{6r} \times 1 - \frac{v}{2t} \times \frac{v}{2n} \right\} \times \frac{P}{n}$

COROL. III.

If C, and D, are of equal ages; then $\frac{1}{2} = \frac{1}{2}$, and $n = v$; whence the probability will become $(1 - \frac{m}{2t} -$

$1 - \frac{v}{3t} \times \frac{v}{2m} - 1 - \frac{v}{2t} \times \frac{v}{6m}$, or) $1 - \frac{m}{2t} - 1 - \frac{v}{3t} \times \frac{3v}{6m} - 1 - \frac{v}{2t} \times \frac{v}{6m}$; that is $(1 - \frac{m}{2t} - 3 - \frac{3v}{3t} \times \frac{v}{6m} - 1 - \frac{v}{2t} \times \frac{v}{6m}$, or) $1 - \frac{m}{2t} - 4 - \frac{3v}{2t} \times \frac{v}{6m}$; that is $1 - \frac{m}{2t} - \frac{2}{3} - \frac{v}{4t} \times \frac{v}{m}$:

And the present worth will be

$\frac{m}{2t} - \frac{1}{2} + \left\{ \frac{m}{6r} \times \frac{v}{2t} - 1 - \frac{v}{2t} \times \frac{1}{2} \right\} \times \frac{P}{m}$, or)

$\frac{m}{2t} - \frac{1}{2} + \frac{1}{6r} \times \frac{3v - 2t}{4t} \times \frac{P}{m}$.

COROL. IV.

If A , B , and C , are of equal ages; then $m = n$, and $t = m = n$; whence the answers will become

$$\left(\frac{1}{2} - \frac{1}{3} - 1 - \frac{v}{2n} \times \frac{vv}{6nn}, \text{ or } \frac{1}{2} - 1 - \frac{v}{2n} \times \frac{vv}{6nn},\right.$$

And

$$\left(\frac{1}{2} - \frac{n}{6r} \times \frac{1}{2} - \frac{v}{6r} \times 1 - \frac{v}{2n} \times \frac{v}{2n} \times \frac{1}{2}, \text{ or } \right.$$

$$\frac{1}{2} - \frac{n}{6r} - \frac{v}{6r} \times \frac{2n - v \times v}{2nn} \times \frac{1}{2}.$$

COROL. V.

If B , C , and D , are of equal ages; then $m = n$, and $m = n = v$; whence the answers will become

$$\left(\frac{1}{2} - \frac{v}{3t} - 1 - \frac{v}{2t} \times \frac{1}{2}, \text{ or } \frac{1}{2} - \frac{v}{4t}, \text{ and } \right.$$

$$\frac{1}{2} - \frac{v}{6r} + \frac{v}{6r} \times \frac{3v - 2t}{4t} \times \frac{1}{2}.$$

A TABLE, shewing the probabilities of survivorship, among four persons of unequal ages; viz. that any two of them, considered as possessors, shall both die before either of the other two; and that either of the last named two persons, considered as expectant, shall die before the other, who is to be considered as the survivor.

SOLUTION.			
If <i>A, B, C,</i> and <i>D,</i> severally represent the names of the young-est, second, third, and eldest, of the four persons; and <i>t, m, n,</i> and <i>v,</i> denote their respective complements of life; then,			
Case	Possessors	Expectant	Survivor
1	A, B, C, D		$\frac{v^3}{12tnm} +$
2	A, C, B, D		$\frac{v^3}{12tnm} +$
3	B, C, A, D		$\frac{v^3}{12tnm} +$
4	A, B, D, C		$\frac{v^3}{4tnm} - \frac{vv}{3mt} +$
5	A, D, B, C	$+\frac{nn}{6tm}$	$\frac{v^3}{12tnm} + \frac{vv}{6mt} -$
6	B, D, A, C	$+\frac{nn}{6tm}$	$\frac{v^3}{12tnm} + \frac{vv}{6mt} -$
7	A, C, D, B	$+\frac{vv}{3tn}$	$\frac{v^3}{4tnm} -$

The Table of the probabilities of survivorship continued.

SOLUTION.

Cafe	Poffitors	Expectant	Survivor					
8	A, L, C	B		$+\frac{n}{2t}$	$-\frac{m}{2tm}$	$-\frac{wv}{6tn}$	$+\frac{wv}{4tnm}$	$\frac{w^3}{4tnm}$
9	C, D, A	B	$+\frac{m}{2t}$	$-\frac{n}{2t}$	$+\frac{nn}{6tm}$	$-\frac{wv}{6tn}$	$+\frac{wv}{12tnm}$	$\frac{w^3}{12tnm}$
10	B, C, D	A			$+\frac{wv}{3nm}$		$-\frac{wv}{4tnm}$	$\frac{w^3}{4tnm}$
11	B, D, C	A		$+\frac{n}{2m}$	$-\frac{nn}{3tm}$	$-\frac{wv}{6nm}$	$+\frac{wv}{12tnm}$	$\frac{w^3}{12tnm}$
12	C, D, B	A	$+\frac{m}{2t}$	$-\frac{n}{2m}$	$+\frac{nn}{6tm}$	$-\frac{wv}{6nm}$	$+\frac{wv}{12tnm}$	$\frac{w^3}{12tnm}$

The sum of all the above probabilities is unity.

QUES.

QUESTION LXII.

The respective ages of four persons, *A*, the youngest; *B*, the second; *C*, the third; and *D*, the eldest; being given: it is required to find the probability of their dying, in any order, that shall be prescribed.

SOLUTION.

The respective probabilities of all the possible varieties, in the order of survivorship among four persons, (whereof any two are supposed to die before either of the other two) were found in the last twelve questions, and their corollaries; and the respective probabilities that (of two persons, both of whose lives are supposed to fail, in a given space of time) either of them shall, within that time, die before the other, were found in question 19: If, therefore, each of the first mentioned probabilities be, severally, multiplied by each of those last mentioned; the results will be the the probabilities now required. To render these processes intelligible, and concise; this question will be divided into twelve cases, referring to the respective questions first above mentioned. And the results of each question, and its corollaries, will be assumed therefrom.

208 MATHEMATICAL

If the solution of the 19th question be applied, to every combination of two persons, in the given four, viz. *A, B, C, and D*; then, the probabilities (during a limited time) of the younger's dying before the elder, and of the elder's dying before the younger, in each of those combinations, will be found in the following table; to which the reader may refer, if any difficulty arises in the following solutions.

Combinations of two persons.	Probability, that the younger will die, before the elder.	Probability, that the elder will die before the younger.
<i>A, B</i>	$\frac{m}{m+t}$	$\frac{t}{m+t}$
<i>A, C</i>	$\frac{n}{n+t}$	$\frac{t}{n+t}$
<i>A, D</i>	$\frac{v}{v+t}$	$\frac{t}{v+t}$
<i>B, C</i>	$\frac{n}{n+m}$	$\frac{m}{n+m}$
<i>B, D</i>	$\frac{v}{v+m}$	$\frac{m}{v+m}$
<i>C, D</i>	$\frac{v}{v+n}$	$\frac{n}{v+n}$

CASE

CASE I. referring to Ques. 50, and its corollaries.

Order of dying.	Ages.	Solution.
A, B, C, D	unequal	$\left\{ \left(\frac{m}{n+t} \times \frac{v^3}{12mn} = \right) \frac{v^3}{m+t \times 12n} \right.$
B, A, C, D		$\left. \left(\frac{t}{n+t} \times \frac{v^3}{12mn} = \right) \frac{v^3}{m+t \times 12n} \right\}$
A, B, C, D B, A, C, D	$A=B$	$\left(\frac{1}{2} \times \frac{v^3}{12mn} = \right) \frac{v^3}{24mn}$
A, B, C, D	$B=C$	$\left\{ \left(\frac{n}{n+t} \times \frac{v^3}{12n} = \right) \frac{v^3}{n+t \times 12n} \right.$
B, A, C, D		$\left. \left(\frac{t}{n+t} \times \frac{v^3}{12n} = \right) \frac{v^3}{n+t \times 12n} \right\}$

CASE

CASE I. continued.

Order of dying	Ages.	Solution.
A, B, C, D	C=D	$\left\{ \left(\frac{m}{m+t} \times \frac{vu}{12im} = \right) \frac{vu}{m+t \times 12t} \right\}$
B, A, C, D		$\left\{ \left(\frac{t}{m+t} \times \frac{vu}{12tm} = \right) \frac{vu}{m+t \times 12m} \right\}$
A, B, C, D B, A, C, D	A=B=C	$\left\{ \left(\frac{1}{3} \times \frac{v^3}{12t^3} = \right) \frac{v^3}{24t^3} \right\}$
A, B, C, D B, A, C, D		$\left\{ \left(\frac{v}{v+t} \times \frac{v}{12t} = \right) \frac{vu}{v+t \times 12t} \right\}$
A, B, C, D B, A, C, D	B=C=D	$\left\{ \left(\frac{t}{v+t} \times \frac{v}{12t} = \right) \frac{v}{v+t \times 12} \right\}$
A, B, C, D B, A, C, D		$\left\{ \text{all equal : } \left(\frac{1}{3} \times \frac{1}{6} \times \frac{1}{6} = \right) \frac{1}{36} \right\}$

CASE

CASE II. referring to quest. 51, and its corollary,

Order of dying.	Ages.	Solution,
A, C, B, D	unequal.	$\left\{ \left(\frac{n}{n+t} \times \frac{v^3}{12mn} \right) = \frac{v^3}{n+t \times 12/m} \right\}$
C, A, B, D		$\left\{ \left(\frac{t}{n+t} \times \frac{v^3}{12mn} \right) = \frac{v^3}{n+t \times 12mn} \right\}$
A, C, B, D	$A=B$	$\left\{ \left(\frac{n}{n+m} \times \frac{v^3}{12mn} \right) = \frac{v^3}{n+m \times 12mn} \right\}$
C, A, B, D		$\left\{ \left(\frac{m}{n+m} \times \frac{v^3}{12mn} \right) = \frac{v^3}{n+m \times 12mn} \right\}$
A, C, B, D	$B=C$	$\left\{ \left(\frac{n}{n+t} \times \frac{v^3}{12tn} \right) = \frac{v^3}{n+t \times 12n} \right\}$
C, A, B, D		$\left\{ \left(\frac{t}{n+t} \times \frac{v^3}{12tn} \right) = \frac{v^3}{n+t \times 12tn} \right\}$

CASE

CASE II. continued.

Solution.

Ages.

Order of dying.

$$\begin{array}{l}
 \left. \begin{array}{l} A, C, B, D \\ C, A, B, D \end{array} \right\} \left. \begin{array}{l} C=D \\ C=D \end{array} \right\} \left(\frac{v}{v+t} \times \frac{v^v}{12tm} = \frac{v^3}{v+t \times 12tm} \right); \\
 \left. \begin{array}{l} A, C, B, D \\ C, A, B, D \end{array} \right\} \left. \begin{array}{l} A=B=C \\ A=B=C \end{array} \right\} \left(\frac{t}{v+t} \times \frac{v^v}{12tm} = \frac{v^v}{v+t \times 12m} \right); \\
 \left. \begin{array}{l} A, C, B, D \\ C, A, B, D \end{array} \right\} \left. \begin{array}{l} A=B=C \\ B=C=D \end{array} \right\} \left(\frac{v}{v+t} \times \frac{v^3}{12t^3} = \frac{v^3}{24t^3} \right); \\
 \left. \begin{array}{l} A, C, B, D \\ C, A, B, D \end{array} \right\} \left. \begin{array}{l} B=C=D \\ B=C=D \end{array} \right\} \left(\frac{v}{v+t} \times \frac{v}{12t} = \frac{v^v}{v+t \times 12t} \right); \\
 \left. \begin{array}{l} A, C, B, D \\ C, A, B, D \end{array} \right\} \left. \begin{array}{l} B=C=D \\ B=C=D \end{array} \right\} \left(\frac{t}{v+t} \times \frac{v}{12t} = \frac{v}{v+t \times 12} \right);
 \end{array}$$

CASE

CASE III. referring to quest. 52, and its Corollary.

Order of dying.	Ages.	Solution.
B, C, A, D	unequal.	$\left(\frac{n}{n+m} \times \frac{v^3}{12fmn} \right) = \frac{v^3}{n+m \times 12fm}$
C, B, A, D		$\left(\frac{m}{n+m} \times \frac{v^3}{12fmn} \right) = \frac{v^3}{n+m \times 12fm}$
B, C, A, D	$A=B$	$\left(\frac{n}{n+m} \times \frac{v^3}{12fmn} \right) = \frac{v^3}{n+m \times 12fm}$
C, B, A, D		$\left(\frac{m}{n+m} \times \frac{v^3}{12fmn} \right) = \frac{v^3}{n+m \times 12fm}$
B, C, A, D	$B=C$	$\left(\frac{1}{2} \times \frac{v^3}{12fmn} \right) = \frac{v^3}{24fmn}$
C, B, A, D		

CASE

CASE III. continued.

Order of dying. Ages Solution.

$$\left. \begin{array}{l} B, C, A, D \\ C, B, A, D \\ B, C, A, D \\ C, B, A, D \\ B, C, A, D \\ C, B, A, D \end{array} \right\} \begin{array}{l} C=D \\ A=B=C; \\ B=C=D; \end{array} \left\{ \begin{array}{l} \left(\frac{v}{v+m} \times \frac{vw}{12tm} \right) = \frac{v^3}{v+m \times 12tm}; \\ \left(\frac{m}{v+m} \times \frac{vw}{12tm} \right) = \frac{vw}{v+m \times 12t}; \\ \left(\frac{1}{2} \times \frac{v}{12n^3} \right) = \frac{v^3}{24n^3}; \\ \left(\frac{1}{2} \times \frac{v}{12t} \right) = \frac{v}{24t}. \end{array} \right.$$

CASE IV. referring to quest. 53, and its corollaries.

$$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\} \begin{array}{l} \text{unequal.} \\ \end{array} \left\{ \begin{array}{l} \left(\frac{m}{m+t} \times \frac{1}{3} - \frac{v}{4n} \right) \times \frac{vw}{mt} = \frac{v}{\frac{1}{3} - \frac{v}{4n}} \times \frac{vw}{m+t \times t}; \\ \left(\frac{t}{m+t} \times \frac{1}{2} - \frac{v}{4n} \right) \times \frac{vw}{mt} = \frac{v}{\frac{1}{2} - \frac{v}{4n}} \times \frac{vw}{m+t \times m}. \end{array} \right.$$

CASE

Order of dying.	Agts.	Solution.
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} A=B; \\ \end{array} \right\}$	$\left(\frac{1}{2} \times \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{nm} \right) = \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{2mn}$
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} B=C; \\ \end{array} \right\}$	$\left(\frac{n}{n+t} \times \frac{1}{2n} - \frac{v}{4n} \times \frac{vu}{nt} \right) = \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{n+t \times t}$
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} C=D; \\ \end{array} \right\}$	$\left(\frac{t}{m+t} \times \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{nt} \right) = \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{n+t \times n}$
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} A=B=C; \\ \end{array} \right\}$	$\left(\frac{m}{n+t} \times \frac{vu}{12mt} \right) = \frac{vu}{n+t \times 12t}$
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} C=D; \\ \end{array} \right\}$	$\left(\frac{t}{m+t} \times \frac{vu}{12mt} \right) = \frac{vu}{m+t \times 12m}$
A, B, D, C. B, A, D, C.	$\left. \begin{array}{l} A=B=C; \\ \end{array} \right\}$	$\left(\frac{1}{2} \times \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{nm} \right) = \frac{1}{2} - \frac{v}{4n} \times \frac{vu}{2mn}$

CASE IV. continued.

Order of dying. Ages.

Solution.

$$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\} B=C=D \left\{ \begin{array}{l} \left(\frac{v}{v+t} \times \frac{v}{12t} = \right) \frac{v^2}{v+t \times 12t} \\ \left(\frac{t}{v+t} \times \frac{v}{12t} = \right) \frac{v}{v+t \times 12} \end{array} \right.$$

CASE V. referring to quest. 54, and its corollaries.

$$\left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} \text{unequal.} \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{v^3}{nn-vv} + \frac{v^3}{2n} \times \frac{1}{v+m} \\ \frac{1}{v+t} \times \frac{v^3}{nn-vv} + \frac{v^3}{2n} \times \frac{1}{v+m} \end{array} \right.$$

$$\left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} A=B \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{v^3}{nn-vv} + \frac{v^3}{2n} \times \frac{1}{v+m} \\ \frac{1}{v+m} \times \frac{v^3}{nn-vv} + \frac{v^3}{2n} \times \frac{1}{v+m} \end{array} \right.$$

CASE

CASE V: continued.

Order of dying.

Solution.

$$\begin{array}{l}
 \left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} B=C \\
 \left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} C=D \\
 \left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} A=R=C
 \end{array}
 \begin{array}{l}
 \frac{v}{v+t} \times \frac{v^3}{m-vv} + \frac{v^3}{2n} \times \frac{1}{6m} \\
 \frac{1}{v+t} \times \frac{v^3}{m-vv} + \frac{v^3}{2n} \times \frac{1}{6n} \\
 \frac{v}{v+t} \times \frac{vv}{12m} \\
 \frac{1}{v+t} \times \frac{vv}{12m} \\
 \frac{v}{v+n} \times \frac{v^3}{m-vv} + \frac{v^3}{2n} \times \frac{1}{6m} \\
 \frac{1}{v+n} \times \frac{v^3}{m-vv} + \frac{v^3}{2n} \times \frac{1}{6n}
 \end{array}$$

CASE V. continued.

Solution.

Order of dying: Ages:

$$\left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} B=C=D \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{v}{12t} \\ \frac{1}{v+t} \times \frac{v}{12} \end{array} \right.$$

CASE VI. referring to quest. 55, and its corollary.

$$\left. \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\} \text{unequal} \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{v^3}{mn-vv+\frac{v^3}{2n}} \times \frac{1}{6tm} \\ \frac{1}{v+m} \times \frac{v^3}{mn-vv+\frac{v^3}{2n}} \times \frac{1}{6t} \end{array} \right.$$

$$\left. \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\} A=B \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{v^3}{mn-vv+\frac{v^3}{2n}} \times \frac{1}{6mn} \\ \frac{1}{v+m} \times \frac{v^3}{mn-vv+\frac{v^3}{2n}} \times \frac{1}{6m} \end{array} \right.$$

CASE

CASE VI. continued.

Order of dying.	Ages.	Solution.			
B, D, A, C	$B=C$	$\frac{v}{v+n}$	$\times \frac{v^3}{nn-uv} + \frac{v^3}{2n}$	$\times \frac{1}{6n}$	
D, B, A, C		$\frac{1}{v+n}$	$\times \frac{v^3}{nn-uv} + \frac{v^3}{2n}$	$\times \frac{1}{6t}$	
B, D, A, C	$C=D$	$\frac{v}{v+m}$	$\times \frac{vu}{12tm}$		
D, B, A, C		$\frac{1}{v+n}$	$\times \frac{v}{12t}$		
B, D, A, C	$A=B=C$	$\frac{v}{v+n}$	$\times \frac{v^3}{nn-uv} + \frac{v^3}{2n}$	$\times \frac{1}{6n}$	
D, B, A, C		$\frac{1}{v+n}$	$\times \frac{v^3}{nn-uv} + \frac{v^3}{2n}$	$\times \frac{1}{6n}$	
B, D, A, C	$B=C=D$	$\frac{v}{v+n}$	$\times \frac{v}{12t}$		
D, B, A, C		$\frac{1}{v+n}$	$\times \frac{v}{12t}$		

CASE VII. referring to quest. 56, and its corollaries.

Order of dying. Ages.

Solution.

$$\begin{array}{l}
 \left. \begin{array}{l} A, C, D, B. \\ C, A, D, B \end{array} \right\} \text{unequal} \quad \left\{ \begin{array}{l} \left(\frac{n}{n+t} \times \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{nt} = \right) \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{n+t} \times t \\ \left(\frac{1}{n+t} \times \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{nt} = \right) \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{n+t} \times n \end{array} \right. \\
 \left. \begin{array}{l} A, C, D, B \\ C, A, D, B \end{array} \right\} A=B \quad \left\{ \begin{array}{l} \left(\frac{n}{n+m} \times \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{nm} = \right) \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{n+m} \\ \left(\frac{m}{n+m} \times \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{nm} = \right) \frac{1}{3} - \frac{v}{4m} \times \frac{vu}{n+m} \end{array} \right. \\
 \left. \begin{array}{l} A, C, D, B \\ C, A, D, B \end{array} \right\} B=C \quad \left\{ \begin{array}{l} \left(\frac{n}{n+t} \times \frac{1}{3} - \frac{v}{4n} \times \frac{vu}{nt} = \right) \frac{1}{3} - \frac{v}{4n} \times \frac{vu}{n+t} \times t \\ \left(\frac{t}{n+t} \times \frac{1}{3} - \frac{v}{4n} \times \frac{vu}{nt} = \right) \frac{1}{3} - \frac{v}{4n} \times \frac{vu}{n+t} \times n \end{array} \right.
 \end{array}$$

CASE

Order of dying.	Ages.	CASE VII continued.	Solution.
A, C, D, B	$C=D$	$\left\{ \left(\frac{v}{v+t} \times \frac{1}{3} - \frac{v}{4m} \times \frac{v}{t} \right) \right.$	$\left. \frac{1}{3} - \frac{v}{4m} \times \frac{v}{v+t} \times \frac{v}{t} \right\}$
C, A, D, B	$C=D$	$\left\{ \left(\frac{t}{v+t} \times \frac{1}{3} - \frac{v}{4m} \times \frac{v}{t} \right) \right.$	$\left. \frac{1}{3} - \frac{v}{4m} \times \frac{v}{v+t} \times \frac{v}{t} \right\}$
C, A, D, B	$A=B=C$	$\left\{ \left(\frac{1}{2} \times \frac{1}{3} - \frac{v}{4m} \times \frac{v}{t} \right) \right.$	$\left. \frac{1}{3} - \frac{v}{4m} \times \frac{v}{t} \right\}$
A, C, D, B	$B=C=D$	$\left\{ \left(\frac{v}{v+t} \times \frac{v}{12t} \right) \right.$	$\left. \frac{v}{v+t} \times \frac{v}{12t} \right\}$
C, A, D, B	$B=C=D$	$\left\{ \left(\frac{t}{v+t} \times \frac{v}{12t} \right) \right.$	$\left. \frac{v}{v+t} \times \frac{v}{12t} \right\}$

CASE VIII. referring to quest. 57, and its corollaries.

Order of dying.	Ages.	Solution.
A, D, C, B	unequal.	$\frac{v}{v+t} \times \frac{1}{\frac{1}{2}} \times \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n} \times \frac{1}{t}$
		$\frac{1}{v+t} \times \frac{1}{\frac{1}{2}} \times \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n}$
A, D, C, B	A=B	$\frac{1}{v+t} \times \frac{1}{\frac{1}{2}} \times \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{3n} \times \frac{1}{t}$
		$\frac{1}{v+t} \times \frac{1}{\frac{1}{2}} \times \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n}$
A, D, C, B	B=C	$\frac{v}{v+t} \times n - 1 - \frac{v}{2n} \times \frac{vv}{n} \times \frac{1}{6t}$
		$\frac{1}{v+t} \times n - 1 - \frac{v}{2n} \times \frac{vv}{n} \times \frac{1}{6}$

CASE

CASE VIII. continued.

Order of dying.	Age.	Solution.
A, D, C, B D, A, C, B	C=D	$\frac{v}{v+f} \times \frac{1}{1} - \frac{v}{4m} \times \frac{v}{f}$
		$\frac{1}{v+f} \times \frac{1}{1} - \frac{v}{4m} \times \frac{v}{f}$
A, D, C, B D, A, C, B	A=B=C	$\frac{v}{v+f} \times \frac{1}{1} - \frac{v}{2n} \times \frac{v}{6m}$
		$\frac{1}{v+f} \times \frac{1}{1} - \frac{v}{2n} \times \frac{v}{6m}$
A, D, C, B D, A, C, B	B=C=D	$\frac{v}{v+f} \times \frac{1}{12f}$
		$\frac{1}{v+f} \times \frac{1}{12f}$

CASE IX. referring to quest. 58, and its corollaries.

Order of dying.	Ages.	Solution.			
C, D, A, B	unequal.	$\frac{v}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$
D, C, A, B		$\frac{n}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$
C, D, A, B	A=B	$\frac{v}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$
D, C, A, B		$\frac{n}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$
C, D, A, B	B=C	$\frac{v}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$
D, C, A, B		$\frac{n}{v+n}$	$\times \frac{m-1}{m}$	$\times \frac{n}{3m}$	$\times \frac{v}{2m}$

CASE

CASE X. referring to quest. 59, and its corollaries.

Order of dying.

Agcs.

Solution.

$$\begin{array}{l}
 \left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} \text{unequal} \quad \left\{ \begin{array}{l} \left(\frac{n}{n+m} \times \frac{1}{3} - \frac{v}{4t} \times \frac{vv}{nm} \right) \frac{v}{\frac{1}{3} - \frac{v}{4t}} \times \frac{vv}{n+m \times m}; \\ \left(\frac{m}{n+m} \times \frac{1}{3} - \frac{v}{4t} \times \frac{vv}{nm} \right) \frac{v}{\frac{1}{3} - \frac{v}{4t}} \times \frac{vv}{n+m \times n}; \end{array} \right. \\
 \left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} \quad A=B \quad \left\{ \begin{array}{l} \left(\frac{n}{n+m} \times \frac{1}{3} - \frac{v}{4t} \times \frac{vv}{nm} \right) \frac{v}{\frac{1}{3} - \frac{v}{4t}} \times \frac{vv}{n+m \times m}; \\ \left(\frac{m}{n+m} \times \frac{1}{3} - \frac{v}{4t} \times \frac{vv}{nm} \right) \frac{v}{\frac{1}{3} - \frac{v}{4t}} \times \frac{vv}{n+m \times n}; \end{array} \right. \\
 \left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} \quad B=C; \left(\frac{1}{3} \times \frac{1}{3} - \frac{v}{4t} \times \frac{vv}{nm} \right) \frac{v}{\frac{1}{3} - \frac{v}{4t}} \times \frac{vv}{2nm};
 \end{array}$$

CASE

CASE X. continued.

Order of Dying.

Ages.

Solution.

$$\left. \begin{array}{l} B, C, D, A \\ C, B, D, A \\ B, C, D, A \\ C, B, D, A \\ B, C, D, A \\ C, B, D, A \end{array} \right\} \begin{array}{l} C=D \\ A=B=C \\ B=C=D \end{array} \left\{ \begin{array}{l} \left(\frac{v}{v+m} \times \frac{1}{2} - \frac{v}{4f} \right) \times \frac{v}{\frac{1}{2} - \frac{v}{m}} = \frac{v}{\frac{1}{2} - \frac{v}{4f}} \times \frac{vv}{v+m} \times \frac{1}{m} \\ \left(\frac{m}{v+m} \times \frac{1}{2} - \frac{v}{4f} \right) \times \frac{v}{\frac{1}{2} - \frac{v}{m}} = \frac{v}{\frac{1}{2} - \frac{v}{4f}} \times \frac{v}{v+m} \\ A=B=C \left(\frac{1}{2} \times \frac{1}{2} - \frac{v}{4f} \right) \times \frac{vv}{m} = \frac{v}{\frac{1}{2} - \frac{v}{4f}} \times \frac{vv}{2mn} \\ B=C=D \left(\frac{1}{2} \times \frac{1}{2} - \frac{v}{4f} \right) \end{array} \right.$$

CASE XI. referring to quest. 60, and its corollaries.

$$\left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} \begin{array}{l} \text{unequal} \end{array} \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{1}{2} - \frac{v}{3f} \times \frac{v}{n-1-\frac{v}{2f}} \times \frac{vv}{6n} \times \frac{1}{m} \\ \frac{1}{v+m} \times \frac{1}{2} - \frac{v}{3f} \times \frac{v}{n-1-\frac{v}{2f}} \times \frac{vv}{6n} \end{array} \right.$$

CASE

CASE XI. continued.

Solution.

Order of Dying. Ages.

$$\begin{array}{l}
 \left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} \begin{array}{l} A=B \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{1}{2} - \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n} \times \frac{1}{m}; \\ \frac{1}{v+m} \times \frac{1}{2} - \frac{n}{3m} \times n - 1 - \frac{v}{2m} \times \frac{vv}{6n}; \end{array} \right. \\
 \left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} \begin{array}{l} B=C \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{1}{2} - \frac{n}{3t} - 1 - \frac{v}{2t} \times \frac{vv}{6nn}; \\ \frac{n}{v+n} \times \frac{1}{2} - \frac{n}{3t} - 1 - \frac{v}{2t} \times \frac{vv}{6nn}; \end{array} \right. \\
 \left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} \begin{array}{l} C=D \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+m} \times 2 - \frac{3v}{2t} \times \frac{v}{6m}; \\ \frac{1}{v+m} \times 2 - \frac{3v}{2t} \times \frac{v}{6}; \end{array} \right.
 \end{array}$$

CASE

CASE XI. continued.

Order of dying.

Solution.

$$\left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} A=B=C \left\{ \frac{v}{v+n} \times \frac{1}{6} - 1 - \frac{v}{2n} \times \frac{qv}{6nm}; \right. \\ \left. \frac{n}{v+n} \times \frac{1}{6} - 1 - \frac{v}{2n} \times \frac{qv}{6nm} \right\} \\ \left. \begin{array}{l} B, D, C, A \\ D, B, C, A \end{array} \right\} B=C=D; \frac{1}{2} \times \frac{1}{2} - \frac{v}{4f}$$

CASE XII. referring to quest. 61, and its corollaries.

$$\left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\} \text{unequal} \left\{ \frac{v}{v+n} \times 1 - \frac{m}{2f} - 1 - \frac{n}{3f} \times \frac{n}{2m} - 1 - \frac{v}{2f} \times \frac{qv}{6nm}; \right. \\ \left. \frac{n}{v+n} \times 1 - \frac{m}{2f} - 1 - \frac{n}{3f} \times \frac{n}{2m} - 1 - \frac{v}{2f} \times \frac{qv}{6nm} \right\}$$

CASE

CASE XII. continued.

Order of dying. Ages. Solution.

$$\begin{array}{l}
 \left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\} A=B \\
 \left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\} B=C \\
 \left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\} C=D; \frac{1}{2} \times 1 - \frac{m}{2f} - \frac{1}{2} - \frac{v}{4f} \times \frac{v}{m}
 \end{array}$$

$$\begin{array}{l}
 \left. \begin{array}{l} \frac{v}{v+n} \times \frac{1}{2} - 1 - \frac{n}{3m} \times \frac{n}{2m} - 1 - \frac{v}{2m} \times \frac{vv}{6nm} \\ \frac{n}{v+n} \times \frac{1}{2} - 1 - \frac{n}{3m} \times \frac{n}{2m} - 1 - \frac{v}{2m} \times \frac{vv}{6nm} \end{array} \right\} \\
 \left. \begin{array}{l} \frac{v}{v+n} \times \frac{1}{2} - 1 - \frac{n}{3f} \times \frac{v}{2f} \times \frac{vv}{6nm} \\ \frac{n}{v+n} \times \frac{1}{2} - 1 - \frac{n}{3f} \times \frac{v}{2f} \times \frac{vv}{6nm} \end{array} \right\}
 \end{array}$$

CASE XII. continued.

Order of dying. Ages.

$$\begin{array}{l}
 C, D, B, A, \left. \begin{array}{l} A=B=C \\ A=B=C \end{array} \right\} \left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{1}{2} - 1 - \frac{v}{2n} \times \frac{vv}{6nn} \\ \frac{n}{v+n} \times \frac{1}{2} - 1 - \frac{v}{2n} \times \frac{vv}{6nn} \end{array} \right. \\
 D, C, B, A, \left. \begin{array}{l} A=B=C \\ A=B=C \end{array} \right\} \left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{1}{2} - 1 - \frac{v}{2n} \times \frac{vv}{6nn} \\ \frac{n}{v+n} \times \frac{1}{2} - 1 - \frac{v}{2n} \times \frac{vv}{6nn} \end{array} \right. \\
 C, D, B, A, \left. \begin{array}{l} B=C=D \\ B=C=D \end{array} \right\} \left\{ \begin{array}{l} \frac{1}{2} \times \frac{1}{2} - \frac{v}{4n} \\ \frac{1}{2} \times \frac{1}{2} - \frac{v}{4n} \end{array} \right.
 \end{array}$$

Solution.

QUESTION

QUESTION LXIII.

The respective ages of 4 persons; *A*, the youngest; *B*, the second; *C*, the third; and *D*, the eldest; being given: it is required, to find the present worth of an estate, or legacy, worth \mathcal{P} pounds, dependent on any order of survivorship, that may be prescribed.

SOLUTION.

If the several present worths, found in the 50th and the eleven following questions, and their corollaries, be multiplied by the same factors, as the probabilities, therein found, have been in the preceding question; the products will be the several present worths required.

Note; when the present worth, resulting from any of those questions, or corollaries, consists of a long expression, and has not a divisor, equal to the numerator of the fraction, into which it is to be multiplied; then the symbol \mathcal{M} , with the addition of the question, or corollary, and page where it may be found, will be wrote instead of that present worth; thus (in case 7) $\frac{n}{n+t} \times \mathcal{M}$, is wrote for,

$$\frac{n}{n+t} \times \left\{ \begin{array}{l} \mathcal{M} - \frac{vv}{6rn} \\ \mathcal{M} - \frac{vv}{6rn} \times 2m-v + \\ \frac{vv}{6rn} \times n-v \times 2 \end{array} \right\} \times \frac{\mathcal{P}}{4mt}$$

CASE I. referring to the present worths found in quest 50, and its corollaries.

Order of dying.	Ages.	Solution.
A, B, C, D	unequal.	$\left\{ \frac{1}{m+t} \times \frac{a}{b} \times \frac{v^p}{4^n} \right\}$
B, A, C, D		$\left\{ \frac{1}{m+t} \times \frac{v}{b} \times \frac{v^p}{4^{mn}} \right\}$
A, B, C, D	$A=B; \frac{1}{2}$	$\left\{ \frac{1}{m+t} \times \frac{v}{b} \times \frac{v^p}{4^{mn}} \right\}$
B, A, C, D		$\left\{ \frac{1}{m+t} \times \frac{v}{b} \times \frac{v^p}{4^{mn}} \right\}$
A, B, C, D	$B=C$	$\left\{ \frac{1}{s+t} \times \frac{a}{b} \times \frac{v^p}{4^n} \right\}$
B, A, C, D		$\left\{ \frac{1}{s+t} \times \frac{v}{b} \times \frac{v^p}{4^{sn}} \right\}$

CASE

CASE I. continued.

Order of dying.	Ages.	Solution.
A, B, C, D	C=D	$\frac{1}{m+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4t}$
B, A, C, D		$\frac{1}{m+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4m}$
A, B, C, D	A=B=C; $\frac{1}{2} \times \frac{v}{6} \times \frac{v}{4t}$	$\frac{1}{m+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4t}$
B, A, C, D		$\frac{1}{m+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4m}$
A, B, C, D	B=C=D	$\frac{1}{v+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4t}$
B, A, C, D		$\frac{1}{v+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4}$
A, B, C, D	all equal; $\frac{1}{2} \times \frac{v}{6} \times \frac{v}{4v}$	$\frac{1}{v+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4v}$
B, A, C, D		$\frac{1}{v+t} \times \frac{v}{6} = \frac{v}{6} \times \frac{v}{4v}$

CASE

CASE II. referring to the present worths, found in quest. 51, and its corollary.

Order of dying.	Ages.	Solution.
A, C, B, D	unequal	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
C, A, B, D	unequal	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
A, C, B, D	A=B	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
C, A, B, D	A=B	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
A, C, B, D	B=C	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
C, A, B, D	B=C	$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$
		$\frac{1}{n+t} \times \frac{a}{b} = \frac{avp}{4nt}$

CASE

CASE II. continued.		Solution.	
Order of dying.	Ages.		
A, C, B, D	C=D	$\frac{v}{v+t}$	$\times \frac{v}{4m}$
		$\frac{1}{v+t}$	$\times \frac{v}{4m}$
C, A, B, D	A=B=C	$\frac{v}{v+t}$	$\times \frac{v}{4n}$
		$\frac{1}{v+t}$	$\times \frac{v}{4n}$
A, C, B, D	B=C=D	$\frac{v}{v+t}$	$\times \frac{v}{4r}$
		$\frac{1}{v+t}$	$\times \frac{v}{4r}$

CASE III. referring to the present worths, found in ques. 52, and its corollary.

Order of dying.	Ages.	Solution.
B, C, A, D	unequal.	$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$
C, B, A, D		$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$
B, C, A, D	$A=B$	$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$
C, B, A, D		$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$
B, C, A, D	$B=C; \frac{1}{2} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$	$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$
C, B, A, D		$\frac{1}{n+m} \times D - \frac{v}{6r} \times \frac{vvD}{4nt};$

CASE

CASE III. continued.

Order of dying. Ages. Solution.

$$\left. \begin{array}{l} B, C, A, D \\ C, B, A, D \end{array} \right\} \begin{array}{l} C=D \\ A=B=C; \frac{1}{2} \times D - \frac{v}{6r} \times \frac{vD}{4r^2}; \\ B=C=D; \frac{1}{2} \times D - \frac{v}{6r} \times \frac{v}{4r^2}. \end{array}$$

CASE IV. referring to the present worths, found in question 53, and its corollaries.

$$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\} \begin{array}{l} \text{unequal.} \\ \frac{1}{m+t} \times \frac{vD}{vD} + \frac{2n-3v}{6rn} \times v \times \frac{vD}{4r^2}; \\ \frac{1}{m+t} \times \frac{vD}{vD} + \frac{2n-3v}{6rn} \times v \times \frac{vD}{4r^2}. \end{array}$$

CASE

CASE IV. continued.

Order of dying.	Ages.	Solution.
$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\}$	$A=B;$	$\frac{1}{2} \times \frac{v}{6r} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4mn}$
$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\}$	$B=C$	$\left\{ \frac{1}{n+t} \times \frac{v}{6r} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4t} \right\}$
$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\}$	$C=D$	$\left\{ \frac{1}{m+t} \times \frac{v}{6r} - \frac{v}{6r} \times \frac{v}{4t} \right\}$
$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\}$	$A=B=C;$	$\frac{1}{2} \times \frac{v}{6r} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4mn}$

CASE

CASE IV. continued.

Order of dying. Ages.

Solution.

$$\left. \begin{array}{l} A, B, D, C \\ B, A, D, C \end{array} \right\} B=C=D \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{v}{6r} \times \frac{p}{4t} \times \frac{p}{4} \\ \frac{1}{v+t} \times \frac{v}{6r} \times \frac{p}{4} \times \frac{p}{4} \end{array} \right.$$

CASE V. referring to the present worths, found in quest. 54, and its corollaries.

$$\left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} \text{unequal.} \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{p}{4t} \times \frac{v}{6r} \times \frac{v}{6r} \times \frac{v}{2n} \times \frac{p}{2mt} \\ \frac{1}{v+t} \times \frac{p}{4t} \times \frac{v}{6r} \times \frac{v}{6r} \times \frac{v}{2n} \times \frac{p}{2mt} \end{array} \right.$$

CASE

CASE V. continued.

Order of dying.	Ages.	Solution.
A, D, B, C	$A=B$	$\left\{ \frac{v}{v+m} \times D - \frac{n}{6r} \times n - D - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2mn} \right\};$
D, A, B, C	$A=B$	$\left\{ \frac{1}{v+m} \times D - \frac{n}{6r} \times n - D - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2m} \right\};$
A, D, B, C	$B=C$	$\left\{ \frac{v}{v+t} \times D - \frac{n}{6r} \times n - D - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2nt} \right\};$
D, A, B, C	$B=C$	$\left\{ \frac{1}{v+t} \times D - \frac{n}{6r} \times n - D - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{P}{2n} \right\};$
A, D, B, C	$C=D$	$\left\{ \frac{v}{v+t} \times D - \frac{v}{6r} \times \frac{vP}{4mt} \right\};$
D, A, B, C	$C=D$	$\left\{ \frac{1}{v+t} \times D - \frac{v}{6r} \times \frac{vP}{4m} \right\};$

CASE V. continued. Solution.

Order of dying. Ages.

$$\begin{array}{l}
 \left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} A=B=C \left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{n}{6r} - \frac{n}{6r} \times n - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{1}{2n}; \\ \frac{1}{v+n} \times \frac{n}{6r} - \frac{n}{6r} \times n - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{1}{2n}; \end{array} \right. \\
 \left. \begin{array}{l} A, D, B, C \\ D, A, B, C \end{array} \right\} B=C=D \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{v}{6r} - \frac{v}{6r} \times \frac{v}{4t}; \\ \frac{1}{v+t} \times \frac{v}{6r} - \frac{v}{6r} \times \frac{v}{4t}. \end{array} \right.
 \end{array}$$

CASE

CASE VI. referring to the present worths, found in Quest. 55, and its corollary.

Order of dying.	Ages.	Solution.
$\left. \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\}$	unequal.	$\left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2mt} \\ \frac{1}{v+m} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2t} \end{array} \right.$
$\left. \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\}$	$A=B$	$\left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2mm} \\ \frac{1}{v+m} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2m} \end{array} \right.$
$\left. \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\}$	$B=C$	$\left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2ln} \\ \frac{1}{v+n} \times \frac{n}{Or} - \frac{n}{Or} \times \frac{v}{Or} - \frac{v}{Or} \times \frac{v}{Or} \times \frac{P}{2t} \end{array} \right.$

M.

CASE

CASE VI. continued.

Order of dying.	Ages.	Solution.
$\left\{ \begin{array}{l} B, \Gamma, A, C \\ D, B, A, C \end{array} \right\}$	$C=D$	$\left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{v}{6r} \times \frac{v}{4mt} \\ \frac{1}{v+m} \times \frac{v}{6r} \times \frac{v}{4t} \end{array} \right\}$
$\left\{ \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\}$	$A=B=C$	$\left\{ \begin{array}{l} \frac{v}{v+n} \times \frac{n}{6r} \times n - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{v}{2n} \\ \frac{1}{v+n} \times \frac{n}{6r} \times n - \frac{v}{6r} \times 1 - \frac{v}{2n} \times v \times \frac{v}{2n} \end{array} \right\}$
$\left\{ \begin{array}{l} B, D, A, C \\ D, B, A, C \end{array} \right\}$	$B=C=D; \frac{1}{2} \times \frac{v}{6r} \times \frac{v}{4t}$	$\frac{v}{4t}$

CASE

CASE VII. referring to the present worths, found in quest. 56, and its corollaries.

Order of dying. Ages.

Solution.

$$\begin{array}{l}
 A, C, D, B \\
 C, A, D, B
 \end{array}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{unequal}
 \left\{
 \begin{array}{l}
 \frac{n}{n+t} \times \mathfrak{M} ; \text{ see quest. 56, fol. 173.} \\
 \frac{1}{n+t} \times \frac{\mathfrak{P}}{4m} \times \left\{ \frac{\frac{vu}{Orn} - \frac{vu}{Orn} \times 2m - \frac{vu}{Orn}}{\frac{vu}{Orn}} \times 2m - v \right\} \\
 + \frac{vu}{Orn} \times n - v \times 2 :
 \end{array}
 \right.$$

$$\begin{array}{l}
 A, C, D, B \\
 C, A, D, B
 \end{array}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ A=B \end{array}
 \left\{
 \begin{array}{l}
 \frac{n}{n+m} \times \mathfrak{M} ; \text{ see Corol. I. quest. 56, fol. 174.} \\
 \frac{1}{n+m} \times \frac{\mathfrak{P}}{4m} \times \left\{ \frac{\frac{vu}{Orn} - \frac{vu}{Orn} \times 2m - \frac{vu}{Orn}}{\frac{vu}{Orn}} \times 2m - v \right\} \\
 + \frac{vu}{Orn} \times n - v \times 2 :
 \end{array}
 \right.$$

CASE VII. continued.

Order of dying.	Ages.	Solution
A, C, D, B	$B=C$	$\left\{ \frac{1}{n+t} \times \frac{1}{v} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4^t}; \right.$
C, A, D, B		$\left. \frac{1}{n+t} \times \frac{1}{v} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4^n}; \right.$
A, C, D, B	$C=D$	$\left\{ \frac{v}{v+t} \times \frac{v}{v} - \frac{v}{6rm} - \frac{v}{6r} \times \frac{2m-v}{2m}; \right.$
C, A, D, B		$\left. \frac{1}{v+t} \times \frac{v}{v} - \frac{v}{6rm} - \frac{v}{6r} \times \frac{2m-v}{2m}; \right.$
A, C, D, B	$A=B=C; \frac{1}{2} \times \frac{1}{v} + \frac{2n-3v}{6rn} \times v \times \frac{v}{4^{2n}};$	$\left. \frac{1}{2} \times \frac{1}{2} \right.$

CASE

CASE VII. continued.

Order of dying.	Ages.	Solution.
A, C, D, B	$B=C=D$	$\frac{v}{v+t} \times D - \frac{v}{or} \times \frac{P}{4t}$
C, A, D, B		$\frac{1}{v+t} \times D - \frac{v}{or} \times \frac{P}{4}$

CASE VIII. referring to the present worths, found in quest. 57, and its corollaries.

A, D, C, B	unequal.	$\frac{v}{v+t} \times \frac{m}{or} - \frac{m}{brn} - \frac{v}{or} \times \frac{2m-v}{2mn}$	$\frac{P}{2t} \times \frac{v}{or}$
D, A, C, B		$\frac{1}{v+t} \times \frac{m}{or} - \frac{m}{brn} - \frac{v}{or} \times \frac{2m-v}{2mn}$	$\frac{P}{2} \times \frac{v}{or}$

CASE VIII. continued.

Order of dying. Ages.

Solution.

$$\begin{array}{l}
 \left. \begin{array}{l} A, D, C, B \\ D, A, C, B \end{array} \right\} \begin{array}{l} A=B \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+m} \times \frac{nn}{6rm} - \frac{v}{6rm} - \frac{v}{6rm} \times \frac{2m-v}{2mn} \times \frac{p}{2m}; \\ \frac{1}{v+m} \times \frac{nn}{6rm} - \frac{v}{6rm} - \frac{v}{6rm} \times \frac{2m-v}{2mn} \times \frac{p}{2}; \end{array} \right. \\
 \left. \begin{array}{l} A, D, C, B \\ D, A, C, B \end{array} \right\} \begin{array}{l} B=C \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{n}{6r} - \frac{v}{6r} - \frac{v}{6r} \times \frac{2n-v}{2n^2} \times \frac{p}{2t}; \\ \frac{1}{v+t} \times \frac{n}{6r} - \frac{v}{6r} - \frac{v}{6r} \times \frac{2n-v}{2n^2} \times \frac{p}{2}; \end{array} \right. \\
 \left. \begin{array}{l} A, D, C, B \\ D, A, C, B \end{array} \right\} \begin{array}{l} C=D \\ \\ \end{array} \left\{ \begin{array}{l} \frac{v}{v+t} \times \frac{vv}{6rm} - \frac{v}{6rm} - \frac{v}{6rm} \times \frac{2m-v}{2m} \times \frac{p}{2t}; \\ \frac{1}{v+t} \times \frac{vv}{6rm} - \frac{v}{6rm} - \frac{v}{6rm} \times \frac{2m-v}{2m} \times \frac{p}{2}; \end{array} \right.
 \end{array}$$

CASE

CASE VIII. continued.

Order of dying,	Ages.	Solution.
A, D, C, B	$\left. \begin{matrix} A=B=C \\ D, A, C, B \end{matrix} \right\}$	$\frac{v}{v+n} \times \frac{n}{br} - \frac{v}{br} \times \frac{2n-v}{2nn} \times v \times \frac{p}{2n};$
D, A, C, B		$\frac{1}{v+n} \times \frac{n}{br} - \frac{v}{br} \times \frac{2n-v}{2nn} \times v \times \frac{p}{2n};$
A, D, C, B	$\left. \begin{matrix} B=C=D \\ D, A, C, B \end{matrix} \right\}$	$\frac{v}{v+f} \times \frac{v}{br} \times \frac{p}{4f};$
D, A, C, B		$\frac{1}{v+f} \times \frac{v}{br} \times \frac{p}{4};$

M 5

CASE

CASE IX. referring to the present worths, found in quest. 58. and its corollaries.

Order of dying	Ages.	Solution.
C, D, A, B	unequal.	$\left\{ \frac{v}{v+n} \times \mathcal{M}; \text{ see quest. 58. fol. 183.} \right.$
D, C, A, B		$\left\{ \frac{n}{v+n} \times \mathcal{M}; \text{ see ditto.} \right.$
C, D, A, B	$A=B$	$\left\{ \frac{v}{v+n} \times \mathcal{M}; \text{ see corol. 1. quest. 58. fol. 185.} \right.$
D, C, A, B		$\left\{ \frac{n}{v+n} \times \mathcal{M}; \text{ see ditto.} \right.$
C, D, A, B	$B=C$	$\left\{ \frac{v}{v+n} \times \mathcal{M} - \frac{n}{6r} - \mathcal{D} - \frac{v}{6r} \times \frac{2n-v}{2nn} \times v \times \frac{\mathcal{D}}{2t}; \right.$
D, C, A, B		$\left\{ \frac{n}{v+n} \times \mathcal{M} - \frac{n}{6r} - \mathcal{D} - \frac{v}{6r} \times \frac{2n-v}{2nn} \times v \times \frac{\mathcal{D}}{2t}; \right.$

CASE

CASE IX, continued.

Solution:

Order of dying. Ages

$$\begin{aligned}
 & \left. \begin{array}{l} C, D, A, B, \\ D, C, A, B \end{array} \right\} \begin{array}{l} C=D; \\ A=B \end{array} \left\{ \begin{array}{l} \frac{1}{2} \times \frac{v}{6r} + \frac{v}{6r} - \frac{3v-2m}{4m} \times \frac{p}{2}; \\ \frac{v}{a+n} \times \frac{n}{6r} - \frac{v}{6r} \times \frac{2n-v}{2m} \times \frac{v}{2n} \times \frac{p}{2}; \\ \frac{1}{a+n} \times \frac{n}{6r} - \frac{v}{6r} \times \frac{2n-v}{2m} \times \frac{v}{2n} \times \frac{p}{2}; \\ \frac{1}{2} \times \frac{v}{6r} - \frac{v}{6r} \times \frac{p}{4} \end{array} \right.
 \end{aligned}$$

CASE X. referring to the present worths, found in quest. 59, and its corollaries.

Order of dying. Ages.

Solution.

$$\left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} \text{unequal} \left\{ \begin{array}{l} \frac{n}{n+m} \times \mathfrak{A}; \\ \frac{m}{n+m} \times \mathfrak{A}; \end{array} \right\} \text{see quest. 59. fol. 190.}$$

$$\left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} \begin{array}{l} A=B \\ A=B \end{array} \left\{ \begin{array}{l} \frac{n}{n+m} \times \mathfrak{A}; \text{ see corol. 1. quest. 59. fol. 191.} \\ \frac{1}{n+m} \times \frac{\mathfrak{B}}{4^m} \times \left\{ \frac{\frac{uvu}{6rm} - \frac{uvu}{6rm} \times 2m - \frac{uvu}{6rm} \times 2m - v}{\frac{uvu}{6rm}} \right\} \\ + \frac{uvu}{6rn} \times n - v \times 2; \end{array} \right.$$

$$\left. \begin{array}{l} B, C, D, A \\ C, B, D, A \end{array} \right\} B=C; \frac{1}{2} \times \mathfrak{A}; \text{ see corol. 2. quest. 59. fol. 192.}$$

CASE

CASE X. continued.

Order of dying.	Ages.	Solution.
B, C, D, A	$C=D$	$\left\{ \frac{v}{v+m} \times \mathcal{A}; \right.$
C, B, D, A		
B, C, D, A	$A=B=C; \frac{1}{2} \times v \mathcal{D} + \frac{2n-3v}{6rn} \times v \times \frac{v \mathcal{D}}{4nn};$	$\left. \frac{m}{v+m} \times \mathcal{A}; \right\}$
C, B, D, A		
B, C, D, A	$B=C=D; \frac{1}{2} \times v \mathcal{D} - \mathcal{D} + \mathcal{A} - \frac{v}{6r} \times \frac{3v-2t}{4t} \times \frac{\mathcal{D}}{v}.$	$\left. \frac{v}{v+m} \times \mathcal{A}; \right\}$
C, B, D, A		

see corol. 3. quest. 59. fol. 192.

CASE XII. referring to the present worths, found in quest. 61, and its corollaries.

Order of dying.

Ages.

Solution.

C, D, B, A	unequal.	$\left\{ \frac{v}{v+n} \right\}$	$\times \mathfrak{A};$	} see quest. 61. fol. 201.
D, C, B, A		$\left\{ \frac{n}{v+n} \right\}$	$\times \mathfrak{B};$	
C, D, B, A	$A = B$	$\left\{ \frac{v}{v+n} \right\}$	$\times \mathfrak{A};$	} see corol. 1. quest. 61. fol. 202.
D, C, B, A		$\left\{ \frac{n}{v+n} \right\}$	$\times \mathfrak{B};$	
C, D, B, A	$B = C$	$\left\{ \frac{v}{v+n} \right\}$	$\times \mathfrak{A};$	} see corol. 2. ditto. 203.
D, C, B, A		$\left\{ \frac{1}{v+n} \right\}$	$\times \mathfrak{B}; \times$	
			$\left\{ \frac{n\mathfrak{A} - \mathfrak{B} + \mathfrak{B} - \frac{n}{6r}}{\frac{v}{6r} \times 1 - \frac{v}{2t} \times \frac{v}{2n}} \right\}$	

CASE

CASE XII. continued.

Order of dying.	Ages.	Solution.
$\left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\}$	$C=D; \frac{1}{2} \times$	$\frac{v}{6r} \times \frac{3v-2t}{4t} \times \frac{D}{m}$
$\left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\}$	$A=B=C$	$\frac{v}{v+n} \times \frac{1}{2} \times \frac{D}{2} \times \frac{v}{6r} \times \frac{2n-v}{2nn} \times v$
$\left. \begin{array}{l} C, D, B, A \\ D, C, B, A \end{array} \right\}$	$B=C=D; \frac{1}{2} \times$	$\frac{v}{6r} \times \frac{3v-2t}{4t} \times \frac{D}{v}$

X 20; see corol. 4. quest. 61. fol. 204.

QUES.

QUESTION LXIV.

The respective ages of four persons; *A*, the youngest; *B*, the second; *C*, the third; and *D* the eldest; being given: it is required to find the probability, that *C* and *D*, the two elder, shall both survive *A* and *B*, the two younger, and also to determine the present value of an estate, or legacy, worth £ pounds, dependent on that survivorship.

SOLUTION.

If *t*, *m*, *n*, and *v*, severally, represent the complements of the lives of *A*, *B*, *C*, and *D*, as in the preceding questions. Then first (by quest. 50.) the probability that *A* and *B*, shall both die before either *C* or *D*; and that *C* and *D*, shall die, in the order in which they stand, is $\frac{v^3}{12tmn}$; and (by quest. 53.) the probability that *A* and *B*, shall both die before either, *D* or *C*; and that *D* and *C* shall die, in the order in which they stand, is $\frac{vv}{3mt} - \frac{v^3}{4tmn}$; now, if these two probabilities be added together, their sum $\left(\frac{vv}{3mt} - \frac{v^3}{6tmn}, \text{ or } 1 - \frac{v}{2n} \times \frac{vv}{3mt} \right)$, will be the probability required.

Note, in the following solutions we shall (for brevity's sake) denote the orders of survivorship, treated off in the 50th, and eleven following questions, by writing, \overline{AB} , $\overline{C,D}$, and similar expressions, instead of saying that *A* and *B*, are both to die, before either *C* or *D*, and that *C* is also to die before *D*, or words similar thereto, adapted to other orders of survivorship.

Secondly; (by quest. 50.) the present worth of an estate, dependent on the order, $\overline{AB, C, D}$, is

$D = \frac{v}{6r} \times \frac{vvD}{4mtn}$; and (by quest. 53) that, of an estate, dependent on the order \overline{AB} , D , C , is

$vD + \frac{2n-3v}{6rn} \times v \times \frac{vD}{4mt}$; therefore their sum

$$\left(C - \frac{v}{6r} \times \frac{vvD}{4mtn} + vD + \frac{2n-3v}{6rn} \times v \times \frac{vD}{4mt} \right), \text{ or}$$

$$C - \frac{v}{6r} \times \frac{v}{n} + vD + \frac{2n-3v}{6rn} \times v \times \frac{vD}{4mt}; \text{ that is}$$

$$\left(\frac{v}{n} + C + vD + \frac{2n-4v}{6r} \times \frac{v}{n} \times \frac{vD}{4mt} \right), \text{ or}$$

$vD + C + \frac{2n-4v}{6r} \times \frac{v}{n} \times \frac{vD}{4mt}$, will be the present worth required.

EXAMPLE.

What is the probability, that C and D (aged 66 and 76) shall both survive, A and B (aged 43 and 54); and what is the present worth of a legacy of 25 \pounds . dependent on that survivorship?

$$\text{Here } \frac{v}{2n} = \left(\frac{10}{40} \right) = \frac{1}{4}; \quad 1 - \frac{1}{4} = \frac{3}{4}; \quad \frac{vv}{3mt} = \left(\frac{10 \cdot 10}{3 \cdot 32 \cdot 43} \right) = \frac{25}{24 \cdot 43};$$

Therefore $\left(\frac{1}{4} \times \frac{25}{24 \cdot 43} \right) = 0,01817$, will be the probability required.

$$\text{Again } vD = 6,215; \quad C = 4,318; \quad \frac{v}{n} = \left(\frac{10}{20} \right) = \frac{1}{2};$$

$$2n-4v = (2 \times 20 - 4 \times 10) = 0; \quad \frac{vD}{4mt} = \frac{10 \times 25}{4 \cdot 32 \cdot 43};$$

And

And $6,215 + \frac{4,318}{2} = 8,374$; Therefore $(8,374 \times \frac{10 \times 25}{4.32.43}) = 0,3804$, will be the present worth required.

COROL. I.

If A , and B , are of equal ages; then $t=m$; and the answers will become $1 - \frac{v}{2n} \times \frac{vv}{3mm}$; and

$$vP + P + \frac{2n-4v}{6r} \times \frac{v}{n} \times \frac{vP}{4mm}.$$

COROL. II.

If B , and C , are of equal ages; then $m=n$; and the answers will become $1 - \frac{v}{2n} \times \frac{vv}{3nt}$, and

$$vP + P + \frac{2n-4v}{6r} \times \frac{v}{n} \times \frac{vP}{4nt}.$$

COROL. III.

If C , and D , are of equal ages; then $vP = P$, and $n=v$; whence the answers will become $(1 - \frac{1}{2} \times \frac{vv}{3mt})$

or) $\frac{vv}{6mt}$, and $(2P - \frac{2v}{6r} \times \frac{vP}{4mt})$ or)

$$P - \frac{v}{6r} \times \frac{vP}{2mt}.$$

COROL.

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$;
and the answers will become $1 - \frac{v}{2n} \times \frac{vv}{3nn}$, and

$$vP + P + \frac{2n-4v}{6r} \times \frac{v}{n} \times \frac{vP}{4nn}.$$

COROL. V.

If B , C , and D , are of equal ages; then $vP = P$,
and $m=n=v$; whence the answers will become $\frac{v}{6t}$,

$$\text{and } P - \frac{v}{6r} \times \frac{P}{2t}.$$

COROL. VI.

If the four lives are of equal ages; then the answers
will be $\frac{1}{6}$, and $P - \frac{v}{6r} \times \frac{P}{2v}$.

QUESTION LXV.

The probability, that B and D shall both survive A
and C ; and the present value of an estate, or legacy,
worth P pounds, dependent on that survivorship, are
required.

SOLU.

SOLUTION.

First. The probability of the order \overline{AC} , B , D , is (by quest. 51) $\frac{v^3}{12mn}$; and that of the order \overline{AC} , D , B , is

(by quest. 56) $\frac{vv}{3tn} - \frac{vv}{4tmn}$; therefore their sum

$\left(\frac{vv}{3tn} - \frac{v^3}{61mn}, \text{ or } 1 - \frac{v}{2m} \times \frac{vv}{3tn}\right)$, will be the probability required.

Secondly; The present worth depending on the order \overline{AC} , B , D is (by quest. 51) $\mathcal{A} - \frac{v}{6r} \times \frac{vv\mathcal{P}}{4mtn}$, and that, on the order \overline{AC} , D , B is (by quest. 56)

$$\left\{ \begin{aligned} & \frac{vv\mathcal{P}}{6rm} \times 2m - \frac{vv\mathcal{P}}{6rn} \times 2m - v \\ & + \frac{vv}{6rn} \times n - v \times 2 \end{aligned} \right\} \times \frac{\mathcal{P}}{4mt};$$

therefore their sum will be the present worth required, viz:

$$\left\{ \begin{aligned} & \frac{vv\mathcal{P}}{6rm} \times 2m - \frac{vv\mathcal{P}}{6rn} \times 2m - v \\ & + \frac{vv}{6rn} \times n - v \times 2 + \mathcal{P} - \frac{v}{6r} \times \frac{vv}{n} \end{aligned} \right\} \times \frac{\mathcal{P}}{4mt}, \text{ or)$$

$$\left\{ \begin{aligned} & \frac{vv\mathcal{P}}{6rm} - \frac{vv\mathcal{P}}{6rn} \times 2m + \mathcal{P} \times \frac{vv}{n} \\ & + \frac{2n - 4v}{6r} \times \frac{vv}{n} + \frac{vv\mathcal{P}}{n} \times v \end{aligned} \right\} \times \frac{\mathcal{P}}{4mt};$$

That is,

$$\begin{aligned} & \left(\frac{v}{2m} - \frac{v}{2m} + \frac{1}{n} - \frac{1}{n} \times \frac{vv}{6r} \times 2m + \frac{vv}{6r} \times v + \frac{2n-4v}{6r} \times \frac{vv}{n} \times \frac{p}{4mt}, \text{ or} \right) \\ & \frac{v}{2m} + \frac{m-n \times v}{mn} \times \frac{v}{6r} \times 2m + \frac{vv}{6r} \times v + \frac{2n-4v}{6r} \times \frac{vv}{n} \times \frac{p}{4mt} \end{aligned}$$

will be the present worth required.

EXAMPLES;

What is the probability, that *B* and *D* (aged 54 and 76) shall both survive *A* and *C* (aged 43 and 66) and what is the present worth of a legacy of 25 £. dependent on that survivorship?

$$\text{Here } \frac{v}{2m} \left(\frac{10}{64} = \right) \frac{5}{32}; 1 - \frac{5}{32} = \frac{27}{32}; \frac{vv}{32} = \left(\frac{10 \cdot 10}{3 \cdot 43 \cdot 20} = \right) \frac{5}{129};$$

Whence $\left(\frac{27}{32} \times \frac{5}{129} = \right) 0.0327$ will be the probability required.

$$\text{Also } \frac{v}{2m} = 6,925; \frac{vv}{6r} = 6,215; \frac{m-n \times v}{mn} = \left(\frac{32-20 \times 10}{32 \times 20} = \right) \frac{12 \cdot 10}{32 \cdot 20} = \frac{3}{16};$$

QUESTION LXVI.

The probability, that A and D shall both survive B and C ; and the present value of an estate, or legacy, worth \mathcal{P} pounds, depending on that survivorship, are required.

SOLUTION.

First, the probability of the order \overline{BC}, A, D is (by quest. 52.) $\frac{v^3}{12tmn}$; and that of the order \overline{BC}, D, A , is

(by quest. 59.) $\frac{vv}{3mn} - \frac{v^3}{4tmn}$; therefore their sum

$\left(\frac{vv}{3mn} - \frac{v^3}{6tmn}, \text{ or } 1 - \frac{v}{2t} \times \frac{vv}{3mn} \right)$, will be the probability required.

Secondly; the present worth depending on the order \overline{BC}, A, D , is (by quest. 52) $\mathcal{P} - \frac{v}{6t} \times \frac{vv\mathcal{P}}{4tmn}$; and that, on the order \overline{BC}, D, A , is (by quest. 59.)

$$\left\{ \frac{v^2 - v^2}{v} + \frac{v^2}{v} \times 2m - \frac{v^2}{v} \times 2l - v + \frac{v^2}{v} \times n - v \times 2 \times \frac{1}{4m} \right\}$$

Therefore, their sum

$$\left\{ \frac{v^2 - v^2}{v} + \frac{1}{4m} \times \left\{ \frac{v^2}{v} \times 2m - \frac{v^2}{v} \times 2l - v + \frac{v^2}{v} \times n - v \times 2 + \frac{v^2}{v} \times \frac{1}{4m} \right\} \right\}, \text{ or}$$

$$\left\{ \frac{v^2 - v^2}{v} + \frac{1}{4m} \times \left\{ \frac{v^2}{v} \times 2m - \frac{v^2}{v} \times 2l - v + \frac{v^2}{v} \times n - v \times 2 + \frac{v^2}{v} \times \frac{1}{4m} \right\} \right\}$$

will be the present worth required.

N

N

EXAMPLE.

What is the probability that *A* and *D* (aged 43 and 76) shall both survive *B* and *C* (aged 54 and 66); and what is the present worth of a legacy of 25*£*. dependent on that survivorship?

$$\text{Here } \frac{v}{2t} = \left(\frac{10}{2.43} = \right) \frac{5}{43}; 1 - \frac{5}{43} = \frac{38}{43}; \frac{vv}{3mn} \\ = \left(\frac{10.10}{3.32.20} = \right) \frac{5}{98};$$

Therefore $\left(\frac{5}{43} \times \frac{5}{98} \pm \frac{38}{43} \times \frac{5}{98} = \right) 0.0461$, will be probability required.

$$\text{Again } v\ddot{x} = 7,228; v\ddot{y} = 6,925; 7,228 - 6,925 \\ = 0.303; \text{ and } \frac{0.303}{10} = 0.0303; v\ddot{y} - \frac{vv}{6rn} \times 2n$$

$$= 411,136; \text{ and } v\ddot{x} - \frac{vv}{6rn} \times 2t - v = 411,426;$$

$$\text{by exam. quest. 59: } \ddot{y} = 4,318; \frac{2n - 3v}{6r} = \left(\frac{10}{6r} = \right)$$

$$1,603; 4,318 + 1,603 = 5,921; \frac{vv}{n} = 5; 5,921 \times 5$$

$$= 29,605; 411,136 - 411,426 + 29,605 = 29,315;$$

$$\frac{29.315}{4.32.43} = 0.0053; 0.0303 + 0.0053 = 0.0356; \text{ and}$$

$(0.0356 \times 25 =) 0.890$, will be the present worth required.

COROLL. I.

If *A*, and *B*, are of equal ages; then $v\ddot{x} = v\ddot{y}$, and $t = n$; whence the answers will become

$$1 - \frac{v}{2n} \times \frac{vv}{3mn}, \text{ and}$$

$$\frac{\ddot{y}}{4mn}$$

$$\frac{p}{4mn} \times \frac{v}{v} - \frac{v}{6rn} \times \frac{v}{v} - \frac{v}{6rn} \times \frac{v}{v} + \frac{2n-3v}{6r} \times \frac{v}{v}.$$

COROL. II.

If B, and C, are of equal ages; then $vB = vC$, and $n = n$; whence the answers will

become $1 - \frac{v}{2f} \times \frac{v}{3rn}$, and

$$\left\{ \frac{v}{v} - \frac{v}{v} \right\} + \frac{1}{4rt} \times \frac{v}{v} - \frac{v}{6rn} \times \frac{v}{v} - \frac{v}{6rn} \times \frac{v}{v} + \frac{2n-3v}{6r} \times \frac{v}{v}.$$

N 3

COROL. III.

If C, and D, are of equal ages; then $vC = vD$, and $n = v$; whence the answers will be-

come $1 - \frac{v}{2f} \times \frac{v}{3rn}$, and

N 3

$$\left\{ \frac{v^2 - v}{v} \times \frac{1}{4vt} \times v - \frac{v}{6r} \times 2m - v - \frac{v}{6r} \times 2t - 3v \right\}$$

COROL. IV.

If A, B, and C, are of equal ages; then $v^2 = v^2$, and $t = m = n$; whence the answers will become $1 - \frac{v}{2n} \times \frac{vv}{3m}$, and

$$\left(\frac{v}{4nn} \times v - \frac{vv}{6r} \times v + v + \frac{2n - 3v}{6r} \times \frac{vv}{n} \text{ or } \frac{v}{4nn} \times v^2 \times v + v + \frac{2n - 4v}{6r} \times \frac{vv}{n}; \text{ that is } \frac{v^3}{4n^3} \times v^2 \times \frac{n}{v} + v + \frac{2n - 4v}{6r} \right)$$

COROL. V.

If B, C, and D are of equal ages; then $v^2 = v^2$ and $m = n = v$; whence the answers will become $1 - \frac{v}{2t} \times \frac{1}{2}$ and

$$\left(\frac{v}{4} \times \frac{v^2 - v}{v} + \frac{1}{4vt} \times v - \frac{v}{6r} \times 4v - 2t \text{ or } \frac{v}{4} \times v^2 - v + v - \frac{v}{6r} \times \frac{4v - 2t}{4t} \right)$$

QUEST.

QUESTION LXVII.

The probability, that *B* and *C* shall both survive *A* and *D*; and the present value of an estate, or legacy, worth *P* pounds, depending on that survivorship are required.

SOLUTION.

First; The probability of the order *AD*, *B*, *C* is (by quest. 54) $\frac{nn}{6tm} - \frac{vv}{6tm} + \frac{v^3}{12ntm}$; and the probability

of the order *AD*, *C*, *B*, is (by quest. 57) $\frac{n}{2t} - \frac{nn}{3tm} -$

$\frac{vv}{6tm} + \frac{v^3}{12ntm}$; therefore their sum $\left(\frac{n}{2t} - \frac{nn}{6tm} -$

$\frac{vv}{6tm} - \frac{vv}{6tm} + \frac{v^3}{6ntm}$ or) $1 - \frac{n}{3m} \times \frac{n}{2t} -$

$\frac{v}{m} + \frac{1}{n} - \frac{v}{nm} \times \frac{vv}{6t}$;

that is $\left(1 - \frac{n}{3m} \times \frac{n}{2t} - \frac{n+m-v}{nm} \times \frac{vv}{6t}\right)$ or)

$1 - \frac{n}{3m} \times \frac{n}{2t} - \frac{n+m-v}{nm} \times \frac{vv}{6t} \times \frac{1}{3}$, will be the probability required.

Secondly; The present worth depending on the order *AD*, *B*, *C*, is (by quest. 54)

$P - \frac{n}{6r} \times \frac{n}{m} - P - \frac{v}{6r} \times 1 - \frac{v}{2n} \times \frac{v}{m} \times \frac{P}{2t}$;

and that on the order *AD*, *C*, *B*, is (by quest. 57)

$P - \frac{nn}{6rm} - P - \frac{v}{6r} \times \frac{2m-v \times v}{2mn} \times \frac{P}{2t}$;

272: MATHEMATICAL

therefore their sum

$$\frac{P}{2r} \times \left\{ P - \frac{n}{6r} \times \frac{n}{m} - P - \frac{v}{6r} \times \frac{2n-v \times v}{2mn} \right\} \text{ or } \\ \frac{P}{2r} \times \left\{ P - \frac{nn}{6rm} + P - \frac{n}{6r} \times \frac{n}{m} - \right. \\ \left. P - \frac{v}{6r} \times \frac{n+m-v}{mn} \times v, \text{ will be the present} \right. \\ \left. \text{worth required.} \right.$$

EXAMPLE.

What is the probability that *B* and *C* (aged 54 and 66) shall both survive *A* and *D* (aged 43 and 76); and what is the present worth of a legacy of 25£. depending on that survivorship?

$$\text{Here } \frac{n}{3m} = \left(\frac{20}{98} = \right) \frac{1}{4.9}; 1 - \frac{1}{4.9} = \frac{4.8}{4.9}; \frac{1.8}{4.9} \times 20 = \\ \left(\frac{19 \times 5}{6} = \right) 15,833; \frac{n+m-v}{nm} = \left(\frac{42}{20.32} = \right) \frac{2.1}{10}; \\ \frac{vv}{3} = \frac{180}{3}; \frac{2.1}{10} \times \frac{180}{3} = \left(\frac{72}{10} = \right) 2,188; 15,833 \\ - 2,188 = 13,645; \text{ and } \left(\frac{13,645}{2 \times 43} = \right) 0,159 \text{ will be} \\ \text{the probability required.}$$

$$\text{Again } \frac{nn}{6rm} = 7,888; P - \frac{n}{6r} = 4,468; \\ \frac{n}{m} = \left(\frac{20}{98} = \right) \frac{1}{4.9}; \\ 4,468 \times \frac{1}{4.9} = 2,7925; P - \frac{v}{6r} = 2,715; \frac{n+m-v}{mn} \\ \times v$$

$\times v = \left(\frac{21 \times 10}{320} = \right) \frac{21}{32}; 2,715 \times \frac{21}{32} = 1,782;$
 $7,888 + 2,793 - 1,782 = 8,899; \frac{P}{2t} = \frac{21}{32};$ therefore
 $(8,899 \times \frac{21}{32} =) 2,587$ will be the present worth re-
 quired.

COROL. I.

If A and B are of equal ages; then $t = m$; and the
 answer will be

$$1 - \frac{n}{3m} \times n - \frac{n+m-v}{nm} \times \frac{vv}{3} \times \frac{1}{2m}; \text{ and} \\
 \frac{P}{2m} \times \left\{ \begin{array}{l} \frac{P}{2m} - \frac{nn}{6rm} \times \frac{P}{2} - \frac{n}{6r} \times \frac{n}{m} \\ - \frac{P}{2} - \frac{v}{6r} \times \frac{n+m-v}{mn} \times v. \end{array} \right.$$

COROL. II.

If B and C are of equal ages; then $n = m$, and
 $m = n$; whence the answers will become

$$\left(1 - \frac{1}{3} \times n - \frac{2n-v}{nn} \times \frac{vv}{3} \times \frac{1}{2t} \text{ or} \right) \\
 2n - \frac{2n-v}{nn} \times vv \times \frac{1}{6t}, \text{ and} \\
 \frac{P}{2t} \times \left(\frac{P}{2} - \frac{n}{6r} \times 2 - \frac{v}{6r} \times \frac{2n-v}{nn} \times v. \right)$$

COROL. III.

If C and D are of equal ages; then $n = m$, and
 $n = v$; whence the answers will become

$$(1 - \frac{v}{3m} \times v - \frac{1}{v} \times \frac{vv}{3} \times \frac{1}{2t} \text{ or})$$

$$\frac{3m-v}{3m} \times v - \frac{v}{3} \times \frac{1}{2t}; \text{ that is } (\frac{3m-v}{m} - 1 \times \frac{v}{6t} \text{ or})$$

$$\frac{2m-v}{m} \times \frac{v}{6t}; \text{ and}$$

$$\frac{1}{2t} \times v - \frac{vv}{6rm} + 1 - \frac{v}{6r} \times \frac{v}{m} - 1, \text{ or}$$

$$\frac{1}{2t} \times v - \frac{vv}{6rm} - 1 - \frac{v}{6r} \times \frac{m-v}{m}$$

COROL. IV.

If A, B, and C are of equal ages; then $m=n=r$, and $t=m=n$; whence the answers will be

$$(2n - \frac{2n-v}{nn} \times vv \times \frac{1}{6n}, \text{ or}) \frac{1}{3} - 2n-v \times \frac{vv}{6n^3};$$

$$\text{and } \frac{1}{2n} \times v - \frac{v}{6r} \times 2 - 1 - \frac{v}{6r} \times \frac{2n-v}{nn} \times v.$$

COROL. V.

If B, C, and D are of equal ages; then $m=n=r$, and $m=n=v$; whence the answers will be

$$(\frac{2v-v}{v} \times \frac{v}{6t} \text{ or}) \frac{v}{6t}; \text{ and } (\frac{1}{2t} \times v - \frac{v}{6r} \times 1 - \frac{v-v}{v})$$

$$\text{or } 1 - \frac{v}{6r} \times \frac{1}{2t}$$

QUES.

QUESTION LXVIII.

The probability that A and C , shall both survive B and D ; and the present value of an estate, or legacy, worth P pounds, depending on that survivorship are required.

SOLUTION.

First, the probability of the order \overline{BD} , A, C , is (by quest. 55.) $\frac{nn}{6tm} - \frac{vv}{6tm} + \frac{v^3}{12tnm}$; and that of the or-

der \overline{BD} , C, A , is (by quest. 60.) $\frac{n}{2m} + \frac{nn}{3tm} - \frac{vv}{6nm} + \frac{v^3}{12tnm}$; th. their sum $\left(\frac{n}{2m} - \frac{nn}{6tm} - \frac{vv}{6tm} - \frac{vv}{6nm} \right.$

$\left. + \frac{v^3}{6tnm} \right)$ or $1 - \frac{n}{3t} \times \frac{n}{2m} - \frac{1}{t} + \frac{1}{n} - \frac{v}{tn} \times \frac{vv}{6m}$;

That is $\left(1 - \frac{n}{3t} \times \frac{n}{2m} - \frac{n+t-v}{tn} \times \frac{vv}{6m} \right)$ or

$1 - \frac{n}{3t} \times n - \frac{n+t-v}{tn} \times \frac{vv}{3} \times \frac{1}{2m}$, will be the probability required.

Secondly; The present worth depending on the order \overline{BD} , A, C , is (by quest. 55.)

$$P - \frac{n}{6r} \times \frac{n}{2tm} - P - \frac{v}{6r} \times \left(1 - \frac{v}{2m} \times \frac{v}{2tm} \right) \times P;$$

and that on the order \overline{BD} , C, A , is (by quest. 60.)

$$\left. \begin{aligned} & \frac{n^2 P}{n} - \frac{n^2 P}{6rm} + \frac{n^2 P}{6rm} \times \frac{1}{2t} \\ & - P - \frac{v}{6r} \times \frac{2t-v \times v}{4tnm} \end{aligned} \right\} \times P;$$

EXAMPLE.

What is the probability that *A* and *C* (aged 43, and 66) shall survive *B* and *D*, (aged 54, and 76) and what is the value of a legacy of 25*£*. dependent on that survivorship.

Here $\frac{n}{3t} = \left(\frac{20}{3.43} = \right) \frac{20}{129}$; $1 - \frac{20}{129} = \frac{109}{129}$;
 $\frac{109}{129} \times 10 = 16,899$; $\frac{n+t-v}{tn} = \frac{53}{860}$; $\frac{vv}{3} =$
 $\frac{109}{3}$; $\frac{53}{860} \times \frac{109}{3} = \left(\frac{53 \times 109}{860} = \right) 2,054$; $16,899 - 2,054$
 $= 14,845$; therefore $\frac{14,845}{2.32} = 0.232$, will be the
 probability required.

Again $\frac{n^2 - n^2}{n} = 0.04735$; $\frac{n^2 - n^2}{6rm} = 7,888$;
 $\frac{n}{6r} \times \frac{n}{m} = 2,792$; $\frac{n}{6r} = 2,715$;
 $\frac{n+t-v}{nm} = \frac{53}{640}$; $\frac{53}{640} \times 10 = \frac{53}{64}$; $2,715 \times \frac{53}{64} =$
 $2,249$; $7,888 + 2,792 - 2,249 = 8,431$; $\frac{8.431}{2 \times 43} =$
 0.09803 ; $0.09803 + 0.04735 = 0.14538$; and 0.14538
 $\times 25 = 3.635$, will be the present worth required.

COROL. I.

If *A* and *B* are of equal ages; then $n^2 = n^2$, and $t=m$; whence the answers will become

$1 - \frac{n}{3m} \times n - \frac{n+m-v}{nm} \times \frac{vv}{3} \times \frac{1}{2m}$, and
 $\frac{1}{2m}$

$$\frac{p}{2m} \times \left\{ \begin{aligned} & \frac{m}{6rm} + \frac{p}{6r} \times \frac{n}{m} \\ & - \frac{p}{6r} \times \frac{m+n-v}{mn} \times v \end{aligned} \right.$$

COROL. II.

If B and C are of equal ages; then $m = n$, and $m = n$; whence the answers will become

$$1 - \frac{n}{3t} \times n - \frac{n+t-v}{tn} \times \frac{vv}{3} \times \frac{1}{2n}, \text{ and}$$

$$p \times \left\{ \begin{aligned} & \frac{n}{6rm} \\ & + \frac{1}{2t} \times \frac{n}{6r} \times 2 - \frac{p}{6r} \times \frac{t+n-v}{mn} \times v \end{aligned} \right.$$

COROL. III.

If C and D are of equal ages; then $p = v$, and $v = v$; whence the answers will become

$$1 - \frac{v}{3t} \times v - \frac{t}{tv} \times \frac{vv}{3} \times \frac{1}{2m}, \text{ or}$$

$$1 - \frac{v}{3t} \times v - \frac{v}{3} \times \frac{1}{2m}; \text{ that is}$$

$$(1 - \frac{v}{3t} \rightarrow \frac{1}{3} \times \frac{v}{2m}, \text{ or}) \frac{2}{3} - \frac{v}{3t} \times \frac{v}{2m} =$$

$$2 - \frac{v}{t} \times \frac{v}{6m}; \text{ and}$$

$$p \times \left\{ \begin{aligned} & \frac{v}{6rm} \\ & + \frac{1}{2t} \times \frac{v}{6rm} - \frac{vv}{6rm} \rightarrow \frac{v}{6r} \times \frac{t-v}{m} \end{aligned} \right.$$

COROL.

COROL. IV.

If A , B , and C , are of equal ages; then $a = b = c = p$, and $t = m = n$; whence the answers will become

$$(1 - \frac{1}{3} \times n - \frac{2n-v}{nn} \times \frac{vv}{3} \times \frac{1}{2n}, \text{ or})$$

$$\frac{2}{3} n - \frac{2n-v}{nn} \times \frac{vv}{3} \times \frac{1}{2n}; \text{ that is}$$

$$(\frac{2n-2n-v}{nn} \times \frac{vv}{6n} \text{ or}) \frac{1}{3} - \frac{2n-v}{6n^2} \times \frac{vv}{6n^2}; \text{ and}$$

$$\frac{p}{2n} \times p - \frac{n}{6r} \times 2 - p - \frac{v}{6r} \times \frac{2n-v}{nn} \times v.$$

COROL. V.

If B , C , and D are of equal ages; then $a = b = c = p$, and $m = n = v$; whence the answers will become

$$\frac{1}{3} - \frac{v}{6t}; \text{ and } p \times \frac{v^2 - p}{v} + \frac{1}{2t} \times p - \frac{v}{6r} \times \frac{2v-t}{v}.$$

QUESTION LXIX.

The probability that A and B , the 2 youngest, shall survive C and D the two eldest; and the present value of an estate, or legacy, worth p pounds, depending on that survivorship, are required.

SOLU.

SOLUTION.

First; The probability of the order \overline{CD} , A, B , is
(by quest. 58) $\frac{m}{2t} - \frac{n}{2t} + \frac{nn}{6mt} - \frac{vv}{6nt} + \frac{v^3}{12nmt}$;

and that, of the order \overline{CD} , B, A is (by quest. 61) $1 - \frac{m}{2t} - \frac{n}{2m} + \frac{nn}{6tm} - \frac{vv}{6nm} + \frac{v^3}{12ntm}$; therefore their

sum $(1 - \frac{n}{2t} - \frac{n}{2m} + \frac{nn}{3tm} - \frac{vv}{6nt} - \frac{vv}{6nm} +$

$\frac{v^3}{6ntm}$, or) $1 - \frac{3}{6t} + \frac{3}{6m} - \frac{2n}{6tm} \times n -$

$\frac{1}{t} + \frac{1}{m} - \frac{v}{tm} \times \frac{vv}{6n}$; that is

$(1 - \frac{3m+3t-2n}{6tm} \times n - \frac{m+t-v}{6tm} \times \frac{vv}{n}, \text{ or})$

$1 - n+t \times 3-2n \times n + m+t-v \times \frac{vv}{n} \times \frac{1}{6tm},$

will be the probability required.

Secondly; the present worth, depending on the order \overline{CD} , A, B , (is by quest. 58)

$P \times \left\{ \frac{m-t}{t} + P - \frac{n}{6r} \times \frac{n}{2tm} \right.$
 $\left. - P - \frac{v}{6r} \times \frac{2m-v}{2n} \times \frac{v}{2tm} \right\}$; and that de-

pending on the order \overline{CD} , B, A , is (by quest. 61)

COROL. IV.

If A , B , and C , are of equal ages; then $m = n = v$, and $t = m = n$; whence the answers will become

$$(1 - 4mn + 2n - v) \times \frac{uv}{n} \times \frac{1}{6mn}$$

$$\text{or } \frac{1}{3} - 2n - v \times \frac{uv}{6mn}; \text{ and}$$

$$\frac{1}{6n} \times \frac{1}{6} - \frac{n}{6r} \times n - \frac{v}{6r} \times \frac{2n - v}{2n} \times v.$$

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$, and $t = m = n = v$; whence the answers will become

$$1 - 4t + v \times \frac{1}{6t}, \text{ and}$$

$$\frac{1}{6t} \times \frac{1}{6} - \frac{1}{6} \times t + \frac{1}{6} - \frac{v}{6r} \times v - \frac{t}{2}.$$

QUESTION LXX.

The respective ages of four persons, A , B , C , and D , being given; it is required to find the probability, that any one of them shall survive the other three?

SOLUTION.

Let t , m , n , and v be the several complements of the lives of A , B , C and D ; and let the orders of dying (whose probabilities were found in the 50th and eleven follow-

following questions) be denoted by \overline{AB} , C , D , or similar expressions, as in the six preceding questions.

CASE I.

If the survivor, D , be elder than the three persons to be survived.

This survivorship will take place, when the four persons die, in either of the following orders, viz. \overline{AB} , C , D ; \overline{AC} , B , D ; or \overline{BC} , A , D :

Now, the probability that the four persons shall die in the order,

$$\overline{AB}, C, D \text{ is (by quest. 50) } \frac{v^3}{12tnm}$$

$$\overline{AC}, B, D \text{ is (by quest. 51) } \frac{v^3}{12tnm}$$

$$\overline{BC}, A, D \text{ is (by quest 52) } \frac{v^3}{12tnm};$$

The sum of which, viz. $\frac{v^3}{4tnm}$, will be the probability required.

EXAMPLE.

If the ages are 43, 54, 66, and 76; then their complements will be 43, 32, 20, and 10; and $\frac{10 \cdot 10 \cdot 10}{4 \cdot 43 \cdot 20 \cdot 32}$

$= \left(\frac{25}{2 \cdot 43 \cdot 32} \right) = 0.009084$ will be the probability required.

COROL. I.

If A and B are of equal ages; then $t = m$; and $\frac{v^3}{4tnm}$ will be the answer.

COROL.

COROL. II.

If B and C are of equal ages; then $m = n$; and $\frac{v^3}{4mn}$ will be the answer.

COROL. III.

If C and D are of equal ages; then $n = v$; and $\frac{v^2}{4tm}$ will be the answer.

COROL. IV.

If A , B and C are of equal ages; then $t = m = n$; and $\frac{v^3}{4n^3}$ will be the answer.

COROL. V.

If B , C , and D are of equal ages; then $m = n = v$; and $\frac{v}{4t}$ will be the answer.

COROL. VI.

If the four lives are of equal ages; then $\frac{1}{4}$ will be the probability required.

CASE II.

If C , the eldest but one, is to survive the other three; This survivorship will take place, when they die in the orders \overline{AB} , D , C ; \overline{AD} , B , C ; and \overline{BD} , A , C :

New

Now the probability of their dying in the order

$$\begin{array}{l} \overline{AB}, D, C \left\{ \begin{array}{l} 53) \\ 54) \\ 55) \end{array} \right. \text{is by quest.} \quad \begin{array}{l} \frac{vv}{3mt} - \frac{v^3}{4nm} \\ \frac{vn}{6tm} - \frac{vv}{6mt} + \frac{v^3}{12nm} \\ \frac{nn}{6tm} - \frac{vv}{6mt} + \frac{v^3}{12nm} \end{array} \end{array}$$

The sum, viz. $\left(\frac{nn}{3tm} - \frac{v^3}{12nm} \right)$ Or $nn - \frac{v^3}{4n} \times \frac{1}{3tm}$, will be the probability required.

EXAMPLE.

The ages being as before, $nn = 400$; $\frac{v^3}{4n} = \frac{25}{2}$;
 $400 - \frac{25}{2} = 387,5$; Th. $\left(\frac{387,5}{3 \cdot 32 \cdot 43} = \right) 0,0939$
 will be the probability required.

COROL. I.

If A and B are of equal ages; then $t = n$; and the answer will be $nn - \frac{v^3}{4n} \times \frac{1}{3nn}$.

COROL. II.

If B and C are of equal ages; then $m = n$; and the answer will be $nn - \frac{v^3}{4n} \times \frac{1}{3tn}$.

COROL.

COROL. III

If C and D are of equal ages; then $n = v$; and the answer will be $(vv - \frac{vv}{4} \times \frac{1}{3^{tm}} \text{ or } \frac{3vv}{4} \times \frac{1}{3^{tm}})$;
That is $\frac{vv}{4^{tm}}$.

COROL. IV.

If A , B , and C , are of equal ages; then $t = m = n$;
and the answer will become $(nn - \frac{v^3}{4n} \times \frac{1}{3^{nn}} \text{ or } \frac{3}{2})$
 $-\frac{v^3}{12n^3}$.

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$;
and the answer will be $\frac{v}{4^t}$.

CASE III

If B , the youngest but one, is to survive the other three.

This survivorship will take place, when they die in the orders, \overline{AC} , D, B ; \overline{AD} , C, B ; and \overline{CD} , A, B :

Now the probability of their dying in the order

$$\left. \begin{array}{l} \overline{AC}, D, B \\ \overline{AD}, C, B \\ \overline{CD}, A, B \end{array} \right\} \begin{array}{l} 56) \\ 57) \\ 58) \end{array} \left. \begin{array}{l} \text{is (by ques.)} \\ \text{is (by ques.)} \\ \text{is (by ques.)} \end{array} \right\} \begin{array}{l} \frac{vv}{3tn} - \frac{v^3}{4tnm} \\ \frac{n}{2t} - \frac{nm}{3tm} - \frac{vv}{6tn} + \frac{v^3}{12tnm} \\ \frac{m}{2t} - \frac{n}{2t} + \frac{nm}{tn} - \frac{vv}{6tn} + \frac{v^3}{12tnm} \end{array}$$

The sum, viz. $\left(\frac{m}{2t} - \frac{nm}{6tm} - \frac{v^3}{12tnm} \right)$ or $\left(\frac{3m}{m} - \frac{nm}{m} - \frac{v^3}{2nm} \right) \times \frac{1}{6t}$;

That is, $3m - nm + \frac{v^3}{2n} \times \frac{1}{m} \times \frac{1}{6t}$, will be the probability required.

EXAMPLE.

The ages being as before, $3m=96$; $nm=400$; $\frac{v^3}{2n} = 25$; $400+25=425$; $\frac{425}{32} =$

13, 281; $96-13$, 281=82, 719;

Therefore $\left(\frac{82,719}{6 \times 43} =\right) 0,321$ will be the probability required.

COROL. I.

If A and B are of equal ages; then $t = m$; and the answer will become $(3m - mn + \frac{v^3}{2n} \times \frac{1}{m} \times \frac{1}{6m}, \text{ or } \frac{1}{2} - mn + \frac{v^3}{2n} \times \frac{1}{6mm}.$

COROL. II.

If B and C are of equal ages; then $m = n$; and the answer will become, $3n - nn + \frac{v^3}{2n} \times \frac{1}{n} \times \frac{1}{6t}.$

COROL. III.

If C and D are of equal ages; then $n = v$; and the answer will become

$$(3m - vv + \frac{vv}{2} \times \frac{1}{m} \times \frac{1}{6t} \text{ or } 3m - \frac{2vv}{2m} \times \frac{2}{6t}.$$

COROL. IV.

If A , B , and C , are of equal ages; then $t = m = n$; and the answer will be $(\frac{1}{2} - mn + \frac{v^3}{2n} \times \frac{1}{6mn}, \text{ or } \frac{1}{2} - \frac{1}{6} - \frac{v^3}{12n^3}; \text{ that is } \frac{1}{2} - \frac{v^3}{12n^3}.$

COROL.

COROL. V.

If B , C , and D , are of equal ages; then $m=n=v$;
and the answer will be $(3v - \frac{3v}{2} \times \frac{1}{6t})$, or

$$\left(\frac{3v}{2} \times \frac{1}{6t} \right) = \frac{v}{4t}$$

CASE IV.

If A , the youngest, is to survive the other three.

This survivorship will take place, when they die in
the orders $\overline{B C}, D, A$; $\overline{B D}, C, A$; and $\overline{C D}, B, A$:

Now the probability of their dying in the order

$$\left. \begin{array}{l} \overline{BC}, D, A \\ \overline{BD}, C, A \\ \overline{CD}, B, A \end{array} \right\} \begin{array}{l} \text{is (by queſt.)} \\ \text{59)} \\ \text{60)} \\ \text{61)} \end{array} \quad \begin{array}{l} \frac{vu}{3nm} - \frac{v^3}{4t^2mn} \\ \frac{vu}{6nm} - \frac{v^3}{12t^2mn} \\ \frac{vu}{6nm} - \frac{v^3}{12t^2mn} \end{array}$$

The ſum, viz. $\left(1 - \frac{m}{2t} - \frac{mn}{6tm} - \frac{v^3}{12t^2mn}\right)$, or $1 - \frac{m}{2t} - \frac{v^3}{2mn} \times \frac{1}{6t}$; that is,

$$1 - 3m - mn + \frac{v^3}{2n} \times \frac{1}{m} \times \frac{1}{6t}, \text{ will be the probability required.}$$

EXAM.

EXAMPLE.

The ages being as before, $3m=96$; $nn=400$; $\frac{v^3}{2n} =$

$$25; 400 + 25 \times \frac{1}{32} = 13,281; 96 + 13,281 = 109,281;$$

$$\text{and } \frac{109,281}{6 \times 43} = 0,424;$$

Therefore $(1 - 0,424 =) 0,576$ will be the probability required.

COROL. I.

If A and B are of equal ages; then $r=m$; and the answer will be $(1 - 3m - nn + \frac{v^3}{2n} \times \frac{1}{m} \times \frac{1}{6m}, \text{ or})$

$$1 - nn + \frac{v^3}{2n} \times \frac{1}{6mm}.$$

COROL. II.

If B and G are of equal ages; then $m=n$; and the answer will be $1 - 3n - nn + \frac{v^3}{2n} \times \frac{1}{n} \times \frac{1}{6t}.$

COROL. III.

If C and D are of equal ages; then $n=v$; and the answer will be $(1 - 3m - vv + \frac{vv}{2} \times \frac{1}{m} \times \frac{1}{6t}, \text{ or})$

$$1 - 3m + \frac{3vv}{2m} \times \frac{1}{6t}.$$

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$;
and the answer will be $(\frac{1}{2} - nn + \frac{v^3}{2n} \times \frac{1}{6nn})$ or

$$\frac{1}{3} - \frac{v^3}{12n^3}.$$

COROL. V.

If B , C , and D , are of equal ages; then $n=s=v$;
and the answer will be $(1 - 3v + \frac{3v}{2} \times \frac{1}{6t})$, or

$$1 - \frac{qv}{12t}; \text{ that is } 1 - \frac{3v}{4t}.$$

The answers, to the above four cases, when the lives are unequal, are as below.

Survived Survivor.

Solution.

$A, B, C.$	$D.$	$\frac{v^3}{4tmn};$
$A, B, D.$	$C.$	$\frac{nn}{3tm} - \frac{v^3}{12tmn};$
$A, C, D.$	$B.$	$\frac{m}{2t} - \frac{nn}{6tm} - \frac{v^3}{12tmn};$
$B, C, D.$	$A.$	$1 - \frac{m}{2t} - \frac{nn}{6tm} - \frac{v^3}{12tmn};$

The sum of which is unity; and the solutions agree with those given, in Mr. *De Moivre's* treatise of annuities on lives.

QUES-

QUESTION. LXXI.

The respective ages of four persons, A , B , C , and D , being given; it is required to find the present value of an estate, or legacy, worth \mathcal{P} pounds, which is to become the property of any one of them, if he survives the other three.

SOLUTION.

Let the complements of life, the values of single lives, and orders of survivorship, be represented by the same symbols, as in the preceding questions.

CASE I.

If the survivor, D , be elder than the three persons to be survived.

Then the present worth of an estate, or legacy, dependent on the order.

$$\overline{AB}, C, D, \text{ is (by quest. 50) } \mathcal{P} - \frac{v}{6r} \times \frac{vv\mathcal{P}}{4mtn},$$

$$\overline{AC}, B, D, \text{ is (by quest. 51) } \mathcal{P} - \frac{v}{6r} \times \frac{vv\mathcal{P}}{4mtn},$$

$$\overline{BC}, A, D, \text{ is (by quest. 52) } \mathcal{P} - \frac{v}{6r} \times \frac{vv\mathcal{P}}{4mtn};$$

Therefore their sum, viz. $\mathcal{P} - \frac{v}{6r} \times \frac{3vv\mathcal{P}}{4mtn}$, will be the present worth required.

EXAMPLE.

D (aged 76) will receive a legacy of 25*l*. if he survives A , B , and C (aged 43, 54, and 66) what is the present value of his interest, in that legacy?

O 4

Here

296 MATHEMATICAL

Here $\frac{v}{6r} = 2,715$; $\frac{3vv}{4mn} = \left(\frac{3.10.10.25}{4.32.43.20} \right)$
 $= \frac{3.5.25}{4.32.43}$;

Therefore $(2,715 \times \frac{3.5.25}{4.32.43} =) 0,185$ will be the present worth required.

COROL. I.

If A and B are of equal ages; then $t=m$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4mn}$;

COROL. II.

If B and C are of equal ages; then $m=n$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4tn}$;

COROL. III.

If C and D are of equal ages; then $n=t$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4mt}$;

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4ttn}$;

COROL.

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$; and the answer will be $a - \frac{v}{6r} \times \frac{3p}{4}$.

COROL. VI.

If the four persons are of equal ages; the answer will be $a - \frac{v}{6r} \times \frac{3p}{4v}$.

CASE II.

If C , the eldest but one, is to survive the other three.

Then the present worth of an estate, or legacy, dependent on the order,

$$\begin{array}{l}
 \overline{AB, D, C} \quad \left\{ \begin{array}{l} 53) \frac{v \cdot 2v}{2} + \frac{2n-3v}{6rn} \times \frac{v \cdot v}{2} \times \frac{v}{2mt}, \\ \overline{AD, B, C} \quad \left\{ \begin{array}{l} 54) \frac{n}{6r} \times n - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{v}{2mt}, \\ \overline{BD, A, C} \quad \left\{ \begin{array}{l} 55) \frac{n}{6r} \times n - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{v}{2mt}; \end{array} \right. \end{array} \right. \\
 \text{is (by quest.)}
 \end{array}$$

$$\begin{array}{l}
 \text{The sum } \frac{v}{2mt} \times \left\{ \begin{array}{l} \frac{v \cdot 2v}{2} + \frac{2n-3v}{6rn} \times \frac{v \cdot v}{2} \\ + \frac{n}{6r} \times 2n - \frac{v}{6r} \times \frac{2n-v}{n} \times v + \frac{2n-v}{2n} \times \frac{v}{6r} \times 2v \end{array} \right\} \\
 \text{or, } \frac{v}{2mt} \left\{ \begin{array}{l} \frac{v \cdot 2v}{2} + \frac{n}{6r} \times 2n + \frac{2n-3v}{2n} \times \frac{v}{6r} \times v \\ - \frac{2n-v}{n} \times v + \frac{4n-2v}{2n} \times \frac{v}{6r} \times v \end{array} \right\}; \text{ that is,}
 \end{array}$$

$$\frac{D}{2mt} \left\{ \begin{aligned} & \frac{vDv}{2} + D - \frac{n}{6r} \times 2n \\ & - D \times \frac{2n-v}{n} \times v + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{aligned} \right\}$$

will be the present worth required.

EXAMPLE.

The ages being as before; $\frac{vDv}{2} = 31,075$;

$$\begin{aligned} D - \frac{n}{6r} &= 4,468; 4,468 \times 2 \times 20 = 178,720; D \\ &= 4,318; \frac{2n-v}{n} \times v = \left(\frac{30-10}{20} = \right) 15; 4,318 \times \\ 15 &= 64,770; \frac{6n-5v}{2n} v = \left(\frac{70-10}{2 \cdot 20} = \right) 17,5; \frac{v}{6r} \\ &= 1,603; 1,603 \times 17,5 = 28,052; 31,075 + 178,720 \\ &- 64,770 + 28,052 = 173,077; \text{ and } \frac{D}{2.m.f} = \frac{25}{2.32.43}; \end{aligned}$$

Therefore $(173,077 \times \frac{25}{2.32.43} =) 1,572$ will be the present worth required.

COROLLARY.

If A , and B , are of equal ages $m = n$;
answer will be

$$\frac{D}{2mm} \times \left\{ \begin{aligned} & \frac{vDv}{2} + D - \frac{n}{6r} \times \\ & - D \times \frac{2n-v}{n} + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{aligned} \right\}$$

Then the present worth of an estate, or legacy, dependent on the order,

$$\begin{array}{l}
 \overline{AB, D, C} \left\{ \begin{array}{l} 53) \frac{vPv}{2} + \frac{2n-3v}{6rn} \times \frac{v^2v}{2} \times \frac{P}{2mt}, \\ 54) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}, \\ 55) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}; \end{array} \right. \\
 \overline{AD, B, C} \left\{ \begin{array}{l} 56) \frac{vPv}{2} + \frac{2n-3v}{6rn} \times \frac{v^2v}{2} \times \frac{P}{2mt}, \\ 57) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}, \\ 58) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}; \end{array} \right. \\
 \overline{BD, A, C} \left\{ \begin{array}{l} 59) \frac{vPv}{2} + \frac{2n-3v}{6rn} \times \frac{v^2v}{2} \times \frac{P}{2mt}, \\ 60) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}, \\ 61) P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v \times \frac{P}{2mt}; \end{array} \right.
 \end{array}$$

$$\begin{array}{l}
 \text{The sum } \frac{P}{2mt} \times \left\{ \begin{array}{l} \frac{vPv}{2} + \frac{2n-3v}{6rn} \times \frac{v^2v}{2} \\ + P - \frac{n}{6r} \times n - P - \frac{v}{6r} \times \frac{2n-v}{2n} \times v + \frac{2n-v}{2n} \times \frac{v}{6r} \times 2v \end{array} \right\} \\
 \text{or, } \frac{P}{2mt} \left\{ \begin{array}{l} \frac{vPv}{2} + P - \frac{n}{6r} \times n + \frac{2n-3v}{2n} \times \frac{v}{6r} \times v \\ - P \times \frac{2n-v}{n} \times v + \frac{4n-2v}{2n} \times \frac{v}{6r} \times v \end{array} \right\}; \text{ that is,}
 \end{array}$$

$$\frac{P}{2m} \left\{ \begin{aligned} & \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ & - P \times \frac{2n-v}{n} \times v + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{aligned} \right\}$$

will be the present worth required.

EXAMPLE.

The ages being as before; $\frac{vPv}{2} = 31,075$;

$$\begin{aligned} P - \frac{n}{6r} &= 4,468; 4,468 \times 2 \times 20 = 178,720; P \\ &= 4,318; \frac{2n-v}{n} \times v = \left(\frac{30-10}{20} = \right) 15; 4,318 \times \\ 15 &= 64,770; \frac{6n-5v}{2n} v = \left(\frac{70-10}{2 \cdot 20} = \right) 17,5; \frac{v}{6r} \\ &= 1,603; 1,603 \times 17,5 = 28,052; 31,075 + 178,720 \\ &- 64,770 + 28,052 = 173,077; \text{ and } \frac{P}{2 \cdot m \cdot t} = \frac{25}{2 \cdot 32 \cdot 43}; \end{aligned}$$

Therefore $(173,077 \times \frac{25}{2 \cdot 32 \cdot 43} =) 1,572$ will be the present worth required.

COROL. I.

If A , and B , are of equal ages; then $t=m$; and the answer will be

$$\frac{P}{2mm} \times \left\{ \begin{aligned} & \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ & - P \times \frac{2n-v}{n} + \frac{6n-5v}{2n} + \frac{vv}{6r} \end{aligned} \right\}$$

COROL. II.

If B and C are of equal ages; then $m=n$; and the answer will be

$$\frac{P}{2tn} \times \left\{ \begin{array}{l} \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ - Pv \times \frac{2n-v}{n} + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{array} \right\}$$

COROL. III.

If C and D are of equal ages; then $P = vP = D$, and $n=v$; whence the answer will become

$$\left(\frac{P}{2tm} \times \left\{ \begin{array}{l} \frac{Pv}{2} + P - \frac{v}{6r} \times 2v \\ - Pv \times \frac{2v-v}{v} + \frac{6v-5v}{2v} \times \frac{vv}{6r}, \text{ or } \end{array} \right\} \right)$$

$$\frac{P}{2tm} \times \frac{Pv}{2} + 2Pv - \frac{2vv}{6r} - Pv + \frac{1}{2} \times \frac{vv}{6r}; \text{ that is,}$$

$$\left(\frac{P}{2tm} \times \frac{3Pv}{2} - \frac{3vv}{2.6r}, \text{ or } \right) \frac{3Pv}{4tm} \times P - \frac{v}{6r}.$$

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$; and the answer will be

$$\frac{P}{2nn} \times \left\{ \begin{array}{l} \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ - Pv \times \frac{2n-v}{n} + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{array} \right\}$$

COROL.

COROL. V.

If B, C, and D, are of equal ages; then $\frac{3D}{4} = \frac{v}{6r}$ and $m = n = v$; whence the answer will be $\frac{3D}{4} \times \frac{v}{6r} = \frac{v}{6r}$

CASE III.

If B, the youngest but one, is to survive the other three.
Then the present worth of an estate, or legacy, dependent on the order,

$$\begin{array}{l}
 \overline{AC, D, B} \quad \left\{ \begin{array}{l} 56) \frac{D}{t} \times \frac{v}{6rm} - \frac{v}{6rm} \times \frac{1}{2} - \frac{v}{6rn} \times \frac{2m-v}{4m} + \frac{v}{6rn} \times \frac{2n-2v}{4m} \\ 57) \frac{D}{t} \times \frac{v}{6rm} - \frac{nn}{6rm} \times \frac{1}{2} - \frac{v}{6r} \times \frac{2m-v}{4mn} \\ 58) \frac{D}{t} \times \frac{v}{6r} - \frac{nn}{6r} \times \frac{1}{2} - \frac{v}{6r} \times \frac{2m-v}{4mn} \end{array} \right. \text{is (by quest.)} \\
 \overline{AD, C, B} \\
 \overline{CD, A, B}
 \end{array}$$

Which

296 MATHEMATICAL

Here $\frac{v}{6r} = 2,715$; $\frac{3vv}{4mtn} = \left(\frac{3 \cdot 10 \cdot 10 \cdot 25}{4 \cdot 32 \cdot 43 \cdot 20} \right)$
 $= \frac{3 \cdot 5 \cdot 25}{4 \cdot 32 \cdot 43}$;

Therefore $(2,715 \times \frac{3 \cdot 5 \cdot 25}{4 \cdot 32 \cdot 43} =) 0,185$ will be the present worth required.

COROL. I.

If A and B are of equal ages; then $t=m$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4mnt}$;

COROL. II.

If B and C are of equal ages; then $m=n$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4tnn}$;

COROL. III.

If C and D are of equal ages; then $n=t$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4mtt}$;

COROL. IV.

If A , B , and C , are of equal ages; then $t=m=n$; and the answer will be $\frac{v}{6r} \times \frac{3vv}{4nnn}$;

COROL.

COROL. V.

If B , C , and D , are of equal ages; then $m = n = v$;
and the answer will be $a - \frac{v}{6r} \times \frac{3p}{4}$.

COROL. VI.

If the four persons are of equal ages; the answer will
be $a - \frac{v}{6r} \times \frac{3p}{4v}$.

CASE II.

If C , the eldest but one, is to survive the other
three.

COROL. II.

If B and C are of equal ages; then $m=n$; and the answer will be

$$\frac{P}{2in} \times \left\{ \begin{array}{l} \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ - Pv \times \frac{2n-v}{n} + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{array} \right\}$$

COROL. III.

If C and D are of equal ages; then $P = vP = D$, and $n=v$; whence the answer will become

$$\left(\frac{P}{2im} \times \left\{ \begin{array}{l} \frac{Pv}{2} + P - \frac{v}{6r} \times 2v \\ - Pv \times \frac{2v-v}{v} + \frac{6v-5v}{2v} \times \frac{vv}{6r}, \text{ or } \end{array} \right\} \right)$$

$$\frac{P}{2im} \times \frac{Pv}{2} + 2Pv - \frac{2vv}{6r} - Pv + \frac{1}{2} \times \frac{vv}{6r}; \text{ that is,}$$

$$\left(\frac{P}{2im} \times \frac{3Pv}{2} - \frac{3vv}{2.6r}, \text{ or } \right) \frac{3Pv}{4im} \times P - \frac{v}{6r}.$$

COROL. IV.

If A , B , and C , are of equal ages; then $i=m=n$; and the answer will be

$$\frac{P}{2nn} \times \left\{ \begin{array}{l} \frac{vPv}{2} + P - \frac{n}{6r} \times 2n \\ - Pv \times \frac{2n-v}{n} + \frac{6n-5v}{2n} \times \frac{vv}{6r} \end{array} \right\}$$

COROL.

COROLLARY V.

If B, C, and D, are of equal ages; then $\frac{B}{C} = \frac{C}{D} = \frac{D}{B}$, and $m = n = v$; whence the answer will be $\frac{3B}{4f} \times \frac{v}{B} - \frac{v}{6r}$

CASE III.

If B, the youngest but one, is to survive the other three.

Then the present worth of an estate, or legacy, dependent on the order,

$$\left. \begin{array}{l} \overline{AC, D, B} \\ \overline{AD, C, B} \\ \overline{CD, A, B} \end{array} \right\} \text{is (by quest.)} \left\{ \begin{array}{l} 56) \frac{B}{f} \times \frac{v}{B} - \frac{v}{6r} \times \frac{1}{2} - \frac{v}{6r} \times \frac{2m-v}{4m} + \frac{v}{6r} \times \frac{2n-2v}{4m}, \\ 57) \frac{B}{f} \times \frac{v}{B} - \frac{v}{6r} \times \frac{1}{2} - \frac{v}{6r} \times \frac{2m-v}{4m} \times \frac{v}{B}, \\ 58) \frac{B}{f} \times \frac{v}{B} - \frac{v}{6r} \times \frac{1}{2} - \frac{v}{6r} \times \frac{n}{2m} - \frac{v}{6r} \times \frac{2m-v}{4m} \times \frac{v}{B}; \end{array} \right.$$

Which

$$\begin{aligned}
 &= 12,620; v \cdot 12 = 6,215; \frac{2m-v}{4} = \left(\frac{54}{4} = \frac{27}{2}\right); 6,215 \times \frac{27}{2} = 83,902; 12 = 4,318; \\
 &2m-v=54; 4,318 \times 54 = 233,172; \frac{v}{6r} = 1,603; \frac{6m-5v}{2} = \left(\frac{192-50}{2} = \right) 71; \\
 &1,603 \times 71 = 113,813; \frac{v}{2m} = \left(\frac{18}{20} = \frac{9}{10}\right); 233,172 - 113,813 = 119,359; 119,359 \times \frac{1}{2} \\
 &= 29,840; 12,620 - 83,902 = -29,840 = -101,172; -101,172 \times \frac{1}{32} = -3,160; 3,084 + \\
 &8,408 - 3,160 = 8,332; \frac{10}{r} = \frac{2\frac{1}{2}}{r}
 \end{aligned}$$

Therefore $(8,332 \times \frac{2\frac{1}{2}}{r} =) 4,844$, will be the present worth required.

COROL. I.

If A and B are of equal ages; then $t=m$; and the answer will be,

$$\left\{ \frac{1}{m} \times \left[\frac{1}{2} \times \frac{1}{m} + \frac{1}{m} \times \frac{1}{m} + \frac{1}{m} \times \frac{1}{m} \right] + \frac{1}{m} \times \left[\frac{1}{2} \times \frac{1}{m} + \frac{1}{m} \times \frac{1}{m} + \frac{1}{m} \times \frac{1}{m} \right] \right\} - \frac{v}{2m} \times \frac{2m-v}{6r} \times \frac{6m-5v}{2}$$

COROL.

$$\left\{ \frac{1}{2} \times \frac{v}{n} + \frac{1}{2} \times \left(\frac{v}{n} - \frac{v}{6r} \right) \times \frac{2n-v}{4} \right\} - \frac{v}{2n} \times \frac{v}{6r} \times \frac{6n-5v}{2}$$

COROLLARY.

If B, C, and D are of equal ages: then $5B = 3C = 2D = v$, and $m = v$; whence the answer will be,

$$\left(\frac{1}{2} \times \frac{v}{4} + \frac{1}{2} \times \frac{v}{6r} + \frac{v}{6r} \times \frac{v}{4} \right) \times \frac{v}{6r} \times \frac{v}{4} \times \frac{v}{6r}$$

That is $\left(\frac{1}{2} \times \frac{v}{4} + \frac{v}{6r} \right) \times \frac{v}{4} \times \frac{v}{6r}$

CASE

CASE IV.

If A , the youngest, is to survive the other three.
Then the present worth of an estate, or legacy, dependent on the order,

$$\begin{array}{l}
 \overline{BC}, D, A \quad \left\{ \begin{array}{l} 59) \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \end{array} \right. \\
 \overline{BD}, C, A \quad \left\{ \begin{array}{l} 60) \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \end{array} \right. \\
 \overline{CD}, B, A \quad \left\{ \begin{array}{l} 61) \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \\ \quad \quad \quad \frac{v}{v} \times \left[\frac{v}{v} - \frac{v}{v} + \frac{v}{v} \times \frac{v}{v} \right] \times \frac{v}{v} \end{array} \right.
 \end{array}$$

is (by question)

which

$$(9,841 + 6,925 \times 16 =) 269,056; \frac{1}{2} \text{ } = 3,836;$$

$$\frac{n}{6r} = 3,205; 3,836 - 3,205 \times 20 = 12,620; \text{ } =$$

$$6,215; \frac{2v}{n} = \left(\frac{2.10}{20} = \right) 1; \frac{2v\text{III}}{n} = (2 =)$$

$$4,318; 6,215 + 4,318 = 10,533; \frac{2f-v}{4} = \left(\frac{76}{4} = \right)$$

$$19; 10,533 \times 19 = 200,127; \frac{v}{6r} = 1,603;$$

$$\frac{6s-5v \times v}{4n} = \left(\frac{208 \times 10}{4 \times 20} = \right) 26; 1,603 \times 26 =$$

$$41,678;$$

$$\text{Now } 210,915 + 269,056 + 12,620 + 41,678 - 200,127 = 334,142;$$

$$\text{And } \frac{334,142}{32 \times 43} = 0,2429; \text{ then } 0,0303 + 0,0473$$

$$+ 0,2429 = 0,3205;$$

Whence $(0,3205 \times 25 =) 8,0125$ will be the present worth required.

COROLL. I.

If A and B are of equal ages; then $v\text{I} = v\text{II}$, $v\text{II} = v\text{III}$, $v\text{III} = v\text{IV}$, and $v\text{IV} = v\text{V}$; whence the answer will become,

$$\left\{ \frac{m}{2} - \frac{m}{2} \times \frac{m}{2} + \frac{m}{2} \times \frac{m}{2} + \frac{m}{2} \times \frac{m}{2} - \frac{m}{2} \times \frac{m}{2} \right\} \times \frac{m}{2} = \frac{m^2}{4}$$

COROL. II.

If, B and C are of equal ages; then $\frac{m}{2} = \frac{m}{2}$, and $m = m$; whence the answer will become,

$$\left\{ \frac{m}{2} - \frac{m}{2} + \frac{m}{2} + \frac{m}{2} - \frac{m}{2} \right\} \times \frac{m}{2} = \frac{m^2}{4}$$

COROL. III.

If C and D are of equal ages; then $\frac{m}{2} = \frac{m}{2}$, and $m = m$; whence the answer will become

✕

$$\left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \times \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \quad ; \text{ that is, } \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\}^2$$

$$\left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \times \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \quad ; \text{ that is, } \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\}^2$$

COROL. V.

If B, C, and D are of equal ages; then $\frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} = \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2}$; and $m = n = v$; whence the answer will become

$$\left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \times \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\} \quad ; \text{ or, } \left\{ \frac{v}{2} + \frac{v}{2} - \frac{v}{6r} \times \frac{v}{2} \right\}^2$$

$$P \times \left\{ \begin{array}{l} \frac{P - P \times \frac{2}{v} + \frac{1}{v}}{\frac{v}{6r} \times \frac{6r - 5v}{4} \times \frac{1}{v}} \end{array} \right.$$

That is,

$$\frac{P}{v} \times \frac{v}{P} - P \times \frac{2}{v} + P - \frac{v}{6r} \times \frac{6r - 5v}{4}$$

_____ gained _____

QUESTION LXXII.

Two persons, B, and C, of given ages, agree to purchase a lease of lands of the clear value of 1*l.* per ann. for their two lives; to be enjoyed equally during their joint lives; and then to belong wholly to the survivor; what part ought each to pay of the purchase money?

SOLUTION.

Each of them is entituled to half the value of the annuity on their joint lives; more the value of the reversion of the whole annuity, for his own life, after the decease of the other.

Now if the value of the single life of the elder be denoted by P , that of the younger by Q ; and the value of their joint lives by R ;

Then half the value of the annuity, on their joint lives, will be $\frac{1}{2} \times R$;

The reversion of the younger life, after the elder, will be $Q - \frac{1}{2} R$ (by quest. 94. vol. 2.)

And the reversion of the elder life, after the younger, will be $P - \frac{1}{2} R$;

Therefore the interest of the younger person will be

$$\frac{1}{2} R - \frac{1}{2} \times \frac{1}{2} R.$$

and that of the elder

$$P - \frac{1}{2} \times \frac{1}{2} R;$$

P 3

That

318. MATHEMATICAL

That is to say, the interest of each person may be found, by subtracting half the value of the joint lives of the two persons, from the value of his own life.

Now if m denote the complement of the younger, and n the complement of the elder life,

Then $\frac{1}{2}P - \frac{1}{2}P - \frac{n}{6r} \times \frac{n}{2m}$ will be the value of their joint lives (by quest. 4.) the half of which (viz.

$\frac{1}{4}P - \frac{1}{4}P - \frac{n}{6r} \times \frac{n}{4m}$) being taken severally, from

$\frac{1}{2}P$ and $\frac{1}{2}P$, the values of the single lives, will leave

$\frac{1}{4}P - \frac{1}{4}P + \frac{1}{2}P - \frac{n}{6r} \times \frac{n}{4m}$ for the interest of the younger,

And $\frac{1}{4}P + \frac{1}{2}P - \frac{n}{6r} \times \frac{n}{4m}$ for the interest of the elder.

EXAMPLE.

If B and C be, severally, aged 54 and 66; then (by exam. 3. quest. 4.) the value of their joint lives is 6,276; Then $10,757 - 3,138 = 7,619$ } will be the sum { younger;
And $7,673 - 3,138 = 4,535$ } to be paid by the { elder.

SCHOLIUM.

If the values of the interest of the above two persons be added together, their sum will be the value of an annuity on the longest of the two lives; and therefore, if the two lives be equal, the interest of each person will be $\frac{1}{2}$ the value of the longest of the two lives.

Again (if the lives be unequal) yet, if the value of the longest of the two lives, and the values of the two single lives be given, the interest of each person may be found thus,

To,

To, or from, half the value of the longest of the two lives, add, or subtract, half the difference of the values of the two single lives; and the sum, or remainder, will be the value of the interest of the younger, or elder person.

For the difference of their interests will appear to be $\frac{1}{2} - \frac{1}{2}$ by subtracting the expressions above given.

QUESTION LXXIII.

Three persons, *A*, *B*, and *C*, of given ages, *A* being younger than *B*, and *B* younger than *C*, have between them a lease of lands, in equal shares; which, on the decease of either of them, is to devolve to the two survivors, share and share alike; and upon the demise of either of them, it is to belong, wholly, to the last survivor for his life; the present value of the interest of each person, in that lease is required?

SOLUTION.

A one of the persons is intitled to $\frac{1}{3}$ of the annuity for the joint lives of the three persons *A B* and *C*; to $\frac{1}{3}$ of the reversion of the same annuity for the two joint lives of *A* and *B*, after the decease of *C*; to $\frac{1}{3}$ of the reversion of the same annuity for the two joint lives of *A* and *C*, after the decease of *B*; and to the whole reversion of the same for his own life, after the longest of the two lives *B* and *C*.

Now, if the values of the single lives of *A*, *B*, and *C* (whose complements are *t*, *m*, and *n*) be called $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{2}$; if the values of the joint lives of any two of them, be denoted by $\frac{1}{2}$, $\frac{1}{2}$, and $\frac{1}{2}$; and the value of the three joint lives by $\frac{1}{2}$:

Then one third of the value of an annuity, on three joint lives, will be $\frac{1}{3} \times \frac{1}{2}$;

Half the reversion of the joint lives of *A* and *B*, after the decease of *C*, will be $\frac{1}{2} \times \overline{s}_{\overline{A+B}|} - \frac{1}{2} \times \overline{s}_{\overline{A}|} \overline{s}_{\overline{B}|}$; and half the reversion of the joint lives of *A* and *C*, after the decease of *B*, will be $\frac{1}{2} \times \overline{s}_{\overline{A+C}|} - \frac{1}{2} \times \overline{s}_{\overline{A}|} \overline{s}_{\overline{C}|}$, by quest. 95. vol. 2.

Also the reversion of the life of *A*, after the longest of *B* and *C* will be $\overline{s}_{\overline{A}|} - \overline{s}_{\overline{A+B}|} - \overline{s}_{\overline{A+C}|} + \overline{s}_{\overline{A+B+C}|}$; by quest. 101. vol. 2.

Whence the interest of *A*, in the annuity, will be $\overline{s}_{\overline{A}|} - \frac{1}{2} \times \overline{s}_{\overline{A+B}|} - \frac{1}{2} \times \overline{s}_{\overline{A+C}|} + \frac{1}{2} \times \overline{s}_{\overline{A+B+C}|}$.

And, by arguing in the same manner, the interest of *B* will be $\overline{s}_{\overline{B}|} - \frac{1}{2} \times \overline{s}_{\overline{A+B}|} - \frac{1}{2} \times \overline{s}_{\overline{B+C}|} + \frac{1}{2} \times \overline{s}_{\overline{A+B+C}|}$;

And, that of *C*, $\overline{s}_{\overline{C}|} - \frac{1}{2} \times \overline{s}_{\overline{A+C}|} - \frac{1}{2} \times \overline{s}_{\overline{B+C}|} + \frac{1}{2} \times \overline{s}_{\overline{A+B+C}|}$.

Therefore, the interest of each person in the lease may be found; by adding one third part of the value of the three joint lives, to the value of his single life; and subtracting therefrom, half the sum of the joint lives of the person (whose interest is required) combined, separately, with each of the other persons concerned.

Hence first. The interest of *A*, the youngest, will by subtracting and adding the values above named, as found in questions 1, 4, and 9 appear to be $\overline{s}_{\overline{A}|} - \frac{1}{2} \overline{s}_{\overline{A+B}|} - \frac{1}{2} \overline{s}_{\overline{A+C}|}$

$$+ \frac{\frac{1}{3}}{4t} \times \overline{s}_{\overline{A+B+C}|} - \frac{m}{6r} \times m - \overline{s}_{\overline{A}|} - \frac{n}{6r} \times \frac{2t-m-n}{3m} \times n.$$

Secondly. The interest of *B*, the middle person will be $\frac{1}{2} \overline{s}_{\overline{B}|} - \frac{1}{2} \overline{s}_{\overline{B+C}|}$

$$+ \frac{\frac{1}{3}}{4t} \times \overline{s}_{\overline{A+B+C}|} - \frac{m}{6r} \times m - \overline{s}_{\overline{B}|} - \frac{n}{6r} \times \frac{2m-t-n}{3m} \times n.$$

Thirdly, the interest of *C*, the eldest, will be

$$\frac{1}{2} \overline{s}_{\overline{C}|} + \overline{s}_{\overline{C}|} - \frac{n}{6r} \times \frac{t+n+n}{12mt} \times n.$$

Now, if we add together the above given expressions of the several interests of *A*, *B*, and *C*, in the lease, viz.

$\overline{s}_{\overline{A+B+C}|}$

[illegible]

GEORGE L. J.

If the two younger persons, A and B , are of equal ages; then $\bar{x} = \bar{y}$, and therefore the values of the several interests will be,

$$\begin{aligned} A & \left\{ \frac{1}{2} \log + \log - \frac{1}{3r} \times \frac{1}{6r} - \frac{1}{2} \log - \frac{1}{6r} \times \frac{1}{12r} \right\} \times \frac{n}{12r} \\ \text{or } B & \left\{ \frac{1}{2} \log + \log - \frac{1}{3r} \times \frac{1}{6r} - \frac{1}{2} \log - \frac{1}{6r} \times \frac{1}{12r} \right\} \times \frac{n}{12r} \\ C & \left\{ \frac{1}{2} \log + \log - \frac{1}{3r} \times \frac{1}{6r} - \frac{1}{2} \log - \frac{1}{6r} \times \frac{1}{12r} \right\} \times \frac{n}{12r} \end{aligned}$$

Secondly, $\frac{1}{2} \times \frac{m}{4t} + \frac{1}{4} \times \frac{m}{6r} - \frac{m}{6r} \times m - \frac{n}{6r} \times \frac{2m-n \times n}{2m}$, will be the va-

lue of B's interest in the lease.

Thirdly, $\frac{n}{6r} \times \frac{2m+n \times n}{8mt}$ will be the value of C's interest.

If the above found values of the interest of A, B, and C, be added together, viz.,

$$A, \quad \frac{1}{2} \times \frac{m}{4t} + \frac{m}{6r} - \frac{m}{6r} \times \frac{m}{4t},$$

$$B, \quad \frac{1}{4} \times \frac{m}{6r} - \frac{m}{6r} \times \frac{m}{4t} - \frac{n}{6r} \times \frac{2m-n \times n}{8mt},$$

$$C, \quad + \frac{n}{6r} - \frac{n}{6r} \times \frac{2m+n \times n}{8mt};$$

The sum, $\frac{1}{2} \times \frac{m}{4t} - \frac{m}{6r} \times \frac{m}{4t} + \frac{n}{6r} - \frac{n}{6r} \times \frac{m}{4mt}$, will be the value of the long-
of the three three lives. See quest. 14.

COROLL. I.

If the two younger lives A and B, are of equal ages; then $\frac{1}{2} = \frac{m}{4t}$, and $t = m$;

There.

Therefore the value of the interests of

$$\begin{aligned}
 A &= \frac{1}{2}P + P - \frac{m}{6r} \times \frac{1}{2}, \\
 B &= \frac{1}{2}P + P - \frac{m}{6r} \times \frac{1}{2} - P - \frac{n}{6r} \times \frac{2m-n}{8mn} \times \frac{1}{2}, \\
 C &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{2m+n}{8mn} \times \frac{1}{2}.
 \end{aligned}$$

COROL. II.

If the two elder persons, B and C, are of equal ages; then $m = n$, and $m = n$; therefore the value of the interests of,

$$\begin{aligned}
 A &= \frac{1}{2}P - \frac{1}{2}P + P - \frac{n}{6r} \times \frac{1}{2} \times \frac{1}{2}, \\
 B &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{1}{2} \times \frac{1}{2}, \\
 C &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{1}{2} \times \frac{1}{2}.
 \end{aligned}$$

COROL.

Secondly, $\frac{1}{4} \times \frac{m}{6r} + \frac{1}{4} \times \frac{m}{6r} \times \frac{m}{6r} - \frac{1}{4} \times \frac{m}{6r} \times \frac{2t-n}{2m} \times \frac{m}{6r}$, will be the value of B's interest.

Thirdly, $\frac{n}{6r} \times \frac{2t-n}{8mt} \times \frac{m}{6r}$; will be the value of C's interest.

Now the sum of these three interests, viz.,

$$\begin{aligned} A's &= \frac{1}{4} \times \frac{m}{6r} + \frac{1}{4} \times \frac{m}{6r} \times \frac{m}{6r} - \frac{1}{4} \times \frac{m}{6r} \times \frac{2t-n}{2m} \times \frac{m}{6r} \\ B's &= \frac{1}{4} \times \frac{m}{6r} + \frac{1}{4} \times \frac{m}{6r} \times \frac{m}{6r} - \frac{1}{4} \times \frac{m}{6r} \times \frac{2t-n}{2m} \times \frac{m}{6r} \\ C's &= \frac{n}{6r} \times \frac{2t-n}{8mt} \times \frac{m}{6r} \end{aligned}$$

That is, $\frac{1}{4} \times \frac{m}{6r} + \frac{1}{4} \times \frac{m}{6r} \times \frac{m}{6r} - \frac{1}{4} \times \frac{m}{6r} \times \frac{2t-n}{2m} \times \frac{m}{6r}$, will be the value of the longest of the three lives, as before.

COROL.

COROL. I.

If the two younger persons, B and A , are of equal ages; then $m = 2n$, and $t = m$; therefore the interest of

$$\begin{aligned} A &= \frac{1}{2}P + \frac{P}{6r} - \frac{m}{6r} \times \frac{1}{4}, \\ B &= \frac{1}{2}P + \frac{P}{6r} - \frac{m}{6r} \times \frac{1}{4} - \frac{n}{6r} \times \frac{2m-n}{8r}, \\ C &= \frac{1}{2}P + \frac{P}{6r} - \frac{n}{6r} \times \frac{2m-n}{8r}. \end{aligned}$$

Which values are the same with those, in corol. i. case 1.

COROL. II.

If the two elder persons, A and C , are of equal ages; then $m = 2n$, and $m = n$; therefore the interest of

$$\begin{aligned} A &= \frac{1}{2}P + \frac{P}{6r} - \frac{n}{6r} \times \frac{1}{4}, \\ B &= \frac{1}{2}P + \frac{P}{6r} - \frac{n}{6r} \times \frac{3n-2t}{8r}, \\ C &= \frac{1}{2}P + \frac{P}{6r} - \frac{n}{6r} \times \frac{2t+n}{8r}. \end{aligned}$$

CASE IV.

If A , be the middle aged person; B , the eldest; and C , the youngest of the three.

Put

Put m , n , and t , for the complements of A , B , and C : then, first,

$$\frac{1}{2} \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{n}{6r} \times \frac{1}{2} \frac{m}{6r}, \text{ will be the value of } A's \text{ interest.}$$

Secondly, $\frac{1}{2} \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{n}{6r} \times \frac{m}{8mt}$, will be the value of B 's interest.

Thirdly,

$\frac{1}{2} \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{2t-n}{4m}$, will be the va-

lue of C 's interest.

Now the sum of these three interests, viz.

$$A's = \frac{1}{2} \frac{m}{6r} - \frac{1}{2} \frac{n}{6r}$$

$$B's = \frac{1}{2} \frac{n}{6r} + \frac{1}{2} \frac{m}{6r}$$

$$C's = \frac{1}{2} \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{2t-n}{4m}$$

That is, $\frac{1}{2} \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{m}{6r} + \frac{1}{2} \frac{n}{6r} \times \frac{2t-n}{4m}$, will be the value of
 the longest of the three lives as before.

COROL. I.

If the two younger persons, A and C , are of equal
 ages; then $r = \frac{1}{2} \frac{m}{6r}$, and $t = m$; therefore the interest
 of

$$A =$$

$$\begin{aligned}
 A &= 22 - \frac{1}{2} 22 \\
 B &= + \frac{1}{2} 22 \\
 C &= + 22 - \frac{n}{6r} \times \frac{n}{4m} \\
 &\quad + 22 - \frac{n}{6r} \times \frac{n}{8m} \\
 &\quad + 22 - \frac{n}{6r} \times \frac{2m-n}{8m} \times n
 \end{aligned}$$

COROL. II.

If the two elder persons, *A* and *B*, are of equal ages; then $22 = 22$, and $n = n$; therefore the interest of

$$\begin{aligned}
 A &= 22 + 22 - \frac{n}{6r} \times \frac{n}{4} \\
 B &= 22 + 22 - \frac{n}{6r} \times \frac{n}{8r} \\
 C &= 22 - 22 + 22 - \frac{n}{6r} \times \frac{n-2r}{8r}
 \end{aligned}$$

CASE V.

If *A* be the eldest person; *B*, the youngest; and *C*, the middle aged person.

Put n, r , and m , for the complements of *A*, *B*, and *C*; then first, $\frac{1}{2} 22 + 22 - \frac{n}{6r} \times \frac{n}{4r}$, will be the value of *A*'s interest. Secondly,

$$22 - \frac{1}{2} 22 + \frac{1}{4r} \times 22 - \frac{m}{6r} \times m - 22 - \frac{n}{6r} \times \frac{2r-n \times n}{2m},$$

will be the value of *B*'s interest.

Thirdly,

$$\begin{aligned}
 A &= \frac{1}{2}P + P - \frac{n}{4m} \times \frac{n}{4m} \\
 B &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{n}{6r} \\
 C &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{n}{6r}
 \end{aligned}$$

COL. II.

If the two elder persons, A and C, are of equal ages; then $\frac{n}{6r} = \frac{n}{4m}$, and $m = n$; therefore the interest of

$$\begin{aligned}
 A &= \frac{1}{2}P + P - \frac{n}{6r} \times \frac{n}{4m} \\
 B &= P - \frac{1}{2}P + P - \frac{n}{6r} \times \frac{3n-2t}{8t} \\
 C &= P - \frac{1}{2}P + P - \frac{n}{6r} \times \frac{2t+n}{8t}
 \end{aligned}$$

CASE VI.

If A be the eldest person; B, the second; and C, the youngest. Put n , m , and t , for the complements of A, B, and C.

Then first; $\frac{1}{2}P + P - \frac{n}{6r} \times \frac{n}{4m}$, will be the value of A's interest.

Se-

Put m , n , and t , for the complements of A , B , and C : then, first,

$$A = \frac{1}{2}r + \frac{1}{2}r - \frac{n}{6r} \times \frac{n}{4m}, \text{ will be the value of } A's \text{ interest.}$$

Secondly, $\frac{1}{2}r + \frac{1}{2}r - \frac{n}{6r} \times \frac{n}{8mt}$, will be the value of $B's$ interest.

Thirdly,

$$A = \frac{1}{2}r + \frac{1}{2}r - \frac{n}{6r} \times \frac{n}{4m} - \frac{n}{6r} \times \frac{2t-n \times n}{4m}, \text{ will be the value of } C's \text{ interest.}$$

Now the sum of these three interests, viz.

$A's = \frac{1}{2}r - \frac{1}{2}r$

$$B's = \frac{1}{2}r - \frac{1}{2}r + \frac{n}{6r} \times \frac{n}{4m}$$

$$C's = \frac{1}{2}r - \frac{1}{2}r + \frac{n}{6r} \times \frac{n}{8mt} + \frac{n}{6r} \times \frac{2t-n \times n}{8mt}$$

That is, $\frac{1}{2}r + \frac{1}{2}r - \frac{n}{6r} \times \frac{n}{2t} + \frac{n}{6r} \times \frac{nn}{4mt}$, will be the value of the longest of the three lives as before.

COROL. I.

If the two younger persons, A and C , are of equal ages; then $r = m$, and $t = m$; therefore the interest

$$A =$$

EXAMPLE.

A (aged 66) has a brother *B* (aged 54) and a daughter *C* (aged 43); they agree to purchase a lease (for the longest of their three lives) of lands, of the clear value of 1*l*. per annum; and it is farther agreed, that the two brothers shall divide the profits, equally, during their joint lives; and, if the younger brother dies first, that the elder shall enjoy the whole, during his life; after which, it is to descend to his daughter; but, if the elder brother dies first, then his daughter and her uncle are to divide the profits, equally, for their joint lives; and the survivor is to possess the whole: What ought each person to contribute toward the purchase?

Answer. *A*'s interest = 4,534,
B's = 6,068,
C's = 4,737;

The sum, = 15,339, which is the value of the longest of the three lives, see exam. to quest. 14.

COROLL. I.

If, *B* and *C*, the two younger, are of equal ages; then $\frac{1}{2} = \frac{1}{2}$, and $t = m$; whence the interest of

Thirdly, $\frac{1}{2}P - \frac{1}{2}P + \frac{1}{4}P - \frac{1}{4}P + \frac{1}{8}P - \frac{1}{8}P + \frac{1}{16}P - \frac{1}{16}P + \frac{1}{32}P - \frac{1}{32}P + \frac{1}{64}P - \frac{1}{64}P + \frac{1}{128}P - \frac{1}{128}P + \frac{1}{256}P - \frac{1}{256}P + \frac{1}{512}P - \frac{1}{512}P + \frac{1}{1024}P - \frac{1}{1024}P + \frac{1}{2048}P - \frac{1}{2048}P + \frac{1}{4096}P - \frac{1}{4096}P + \frac{1}{8192}P - \frac{1}{8192}P + \frac{1}{16384}P - \frac{1}{16384}P + \frac{1}{32768}P - \frac{1}{32768}P + \frac{1}{65536}P - \frac{1}{65536}P + \frac{1}{131072}P - \frac{1}{131072}P + \frac{1}{262144}P - \frac{1}{262144}P + \frac{1}{524288}P - \frac{1}{524288}P + \frac{1}{1048576}P - \frac{1}{1048576}P + \frac{1}{2097152}P - \frac{1}{2097152}P + \frac{1}{4194304}P - \frac{1}{4194304}P + \frac{1}{8388608}P - \frac{1}{8388608}P + \frac{1}{16777216}P - \frac{1}{16777216}P + \frac{1}{33554432}P - \frac{1}{33554432}P + \frac{1}{67108864}P - \frac{1}{67108864}P + \frac{1}{134217728}P - \frac{1}{134217728}P + \frac{1}{268435456}P - \frac{1}{268435456}P + \frac{1}{536870912}P - \frac{1}{536870912}P + \frac{1}{1073741824}P - \frac{1}{1073741824}P + \frac{1}{2147483648}P - \frac{1}{2147483648}P + \frac{1}{4294967296}P - \frac{1}{4294967296}P + \frac{1}{8589934592}P - \frac{1}{8589934592}P + \frac{1}{17179869184}P - \frac{1}{17179869184}P + \frac{1}{34359738368}P - \frac{1}{34359738368}P + \frac{1}{68719476736}P - \frac{1}{68719476736}P + \frac{1}{137438953472}P - \frac{1}{137438953472}P + \frac{1}{274877906944}P - \frac{1}{274877906944}P + \frac{1}{549755813888}P - \frac{1}{549755813888}P + \frac{1}{1099511627776}P - \frac{1}{1099511627776}P + \frac{1}{2199023255552}P - \frac{1}{2199023255552}P + \frac{1}{4398046511104}P - \frac{1}{4398046511104}P + \frac{1}{8796093022208}P - \frac{1}{8796093022208}P + \frac{1}{17592186044416}P - \frac{1}{17592186044416}P + \frac{1}{35184372088832}P - \frac{1}{35184372088832}P + \frac{1}{70368744177664}P - \frac{1}{70368744177664}P + \frac{1}{140737488355328}P - \frac{1}{140737488355328}P + \frac{1}{281474976710656}P - \frac{1}{281474976710656}P + \frac{1}{562949953421312}P - \frac{1}{562949953421312}P + \frac{1}{1125899906842624}P - \frac{1}{1125899906842624}P + \frac{1}{2251799813685248}P - \frac{1}{2251799813685248}P + \frac{1}{4503599627370496}P - \frac{1}{4503599627370496}P + \frac{1}{9007199254740992}P - \frac{1}{9007199254740992}P + \frac{1}{18014398509481984}P - \frac{1}{18014398509481984}P + \frac{1}{36028797018963968}P - \frac{1}{36028797018963968}P + \frac{1}{72057594037927936}P - \frac{1}{72057594037927936}P + \frac{1}{144115188075855872}P - \frac{1}{144115188075855872}P + \frac{1}{288230376151711744}P - \frac{1}{288230376151711744}P + \frac{1}{576460752303423488}P - \frac{1}{576460752303423488}P + \frac{1}{1152921504606846976}P - \frac{1}{1152921504606846976}P + \frac{1}{2305843009213693952}P - \frac{1}{2305843009213693952}P + \frac{1}{4611686018427387904}P - \frac{1}{4611686018427387904}P + \frac{1}{9223372036854775808}P - \frac{1}{9223372036854775808}P + \frac{1}{18446744073709551616}P - \frac{1}{18446744073709551616}P + \frac{1}{36893488147419103232}P - \frac{1}{36893488147419103232}P + \frac{1}{73786976294838206464}P - \frac{1}{73786976294838206464}P + \frac{1}{147573952589676412928}P - \frac{1}{147573952589676412928}P + \frac{1}{295147905179352825856}P - \frac{1}{295147905179352825856}P + \frac{1}{590295810358705651712}P - \frac{1}{590295810358705651712}P + \frac{1}{1180591620717411303424}P - \frac{1}{1180591620717411303424}P + \frac{1}{2361183241434822606848}P - \frac{1}{2361183241434822606848}P + \frac{1}{4722366482869645213696}P - \frac{1}{4722366482869645213696}P + \frac{1}{9444732965739290427392}P - \frac{1}{9444732965739290427392}P + \frac{1}{18889465931478580854784}P - \frac{1}{18889465931478580854784}P + \frac{1}{37778931862957161709568}P - \frac{1}{37778931862957161709568}P + \frac{1}{75557863725914323419136}P - \frac{1}{75557863725914323419136}P + \frac{1}{151115727451828646838272}P - \frac{1}{151115727451828646838272}P + \frac{1}{302231454903657293676544}P - \frac{1}{302231454903657293676544}P + \frac{1}{604462909807314587353088}P - \frac{1}{604462909807314587353088}P + \frac{1}{1208925819614629174706176}P - \frac{1}{1208925819614629174706176}P + \frac{1}{2417851639229258349412352}P - \frac{1}{2417851639229258349412352}P + \frac{1}{4835703278458516698824704}P - \frac{1}{4835703278458516698824704}P + \frac{1}{9671406556917033397649408}P - \frac{1}{9671406556917033397649408}P + \frac{1}{19342813113834066795298816}P - \frac{1}{19342813113834066795298816}P + \frac{1}{38685626227668133590597632}P - \frac{1}{38685626227668133590597632}P + \frac{1}{77371252455336267181195264}P - \frac{1}{77371252455336267181195264}P + \frac{1}{154742504910672534362390528}P - \frac{1}{154742504910672534362390528}P + \frac{1}{309485009821345068724781056}P - \frac{1}{309485009821345068724781056}P + \frac{1}{618970019642690137449562112}P - \frac{1}{618970019642690137449562112}P + \frac{1}{1237940039285380274899124224}P - \frac{1}{1237940039285380274899124224}P + \frac{1}{2475880078570760549798248448}P - \frac{1}{2475880078570760549798248448}P + \frac{1}{4951760157141521099596496896}P - \frac{1}{4951760157141521099596496896}P + \frac{1}{9903520314283042199192993792}P - \frac{1}{9903520314283042199192993792}P + \frac{1}{19807040628566084398385987584}P - \frac{1}{19807040628566084398385987584}P + \frac{1}{39614081257132168796771975168}P - \frac{1}{39614081257132168796771975168}P + \frac{1}{79228162514264337593543950336}P - \frac{1}{79228162514264337593543950336}P + \frac{1}{158456325028528675187087900672}P - \frac{1}{158456325028528675187087900672}P + \frac{1}{316912650057057350374175801344}P - \frac{1}{316912650057057350374175801344}P + \frac{1}{633825300114114700748351602688}P - \frac{1}{633825300114114700748351602688}P + \frac{1}{1267650600228229401496703205376}P - \frac{1}{1267650600228229401496703205376}P + \frac{1}{2535301200456458802993406410752}P - \frac{1}{2535301200456458802993406410752}P + \frac{1}{5070602400912917605986812821504}P - \frac{1}{5070602400912917605986812821504}P + \frac{1}{10141204801825835211973625643008}P - \frac{1}{10141204801825835211973625643008}P + \frac{1}{20282409603651670423947251286016}P - \frac{1}{20282409603651670423947251286016}P + \frac{1}{40564819207303340847894502572032}P - \frac{1}{40564819207303340847894502572032}P + \frac{1}{81129638414606681695789005144064}P - \frac{1}{81129638414606681695789005144064}P + \frac{1}{162259276829213363391578010288128}P - \frac{1}{162259276829213363391578010288128}P + \frac{1}{324518553658426726783156020576256}P - \frac{1}{324518553658426726783156020576256}P + \frac{1}{649037107316853453566312041152512}P - \frac{1}{649037107316853453566312041152512}P + \frac{1}{1298074214633706907132624082305024}P - \frac{1}{1298074214633706907132624082305024}P + \frac{1}{2596148429267413814265248164610048}P - \frac{1}{2596148429267413814265248164610048}P + \frac{1}{5192296858534827628530496329220096}P - \frac{1}{5192296858534827628530496329220096}P + \frac{1}{10384593717069655257060992658440192}P - \frac{1}{10384593717069655257060992658440192}P + \frac{1}{20769187434139310514121985316880384}P - \frac{1}{20769187434139310514121985316880384}P + \frac{1}{41538374868278621028243970633760768}P - \frac{1}{41538374868278621028243970633760768}P + \frac{1}{83076749736557242056487941267521536}P - \frac{1}{83076749736557242056487941267521536}P + \frac{1}{166153499473114484112975882535043072}P - \frac{1}{166153499473114484112975882535043072}P + \frac{1}{332306998946228968225951765070086144}P - \frac{1}{332306998946228968225951765070086144}P + \frac{1}{664613997892457936451903530140172288}P - \frac{1}{664613997892457936451903530140172288}P + \frac{1}{1329227995784915872903807060280344576}P - \frac{1}{1329227995784915872903807060280344576}P + \frac{1}{2658455991569831745807614120560689152}P - \frac{1}{2658455991569831745807614120560689152}P + \frac{1}{5316911983139663491615228241121378304}P - \frac{1}{5316911983139663491615228241121378304}P + \frac{1}{10633823966279326983230456482242756608}P - \frac{1}{10633823966279326983230456482242756608}P + \frac{1}{21267647932558653966460912964485513216}P - \frac{1}{21267647932558653966460912964485513216}P + \frac{1}{42535295865117307932921825928971026432}P - \frac{1}{42535295865117307932921825928971026432}P + \frac{1}{85070591730234615865843651857942052864}P - \frac{1}{85070591730234615865843651857942052864}P + \frac{1}{170141183460469231731687303715884105728}P - \frac{1}{170141183460469231731687303715884105728}P + \frac{1}{340282366920938463463374607431768211456}P - \frac{1}{340282366920938463463374607431768211456}P + \frac{1}{680564733841876926926749214863536422912}P - \frac{1}{680564733841876926926749214863536422912}P + \frac{1}{1361129467683753853853498429727072845824}P - \frac{1}{1361129467683753853853498429727072845824}P + \frac{1}{2722258935367507707706996859454145691648}P - \frac{1}{2722258935367507707706996859454145691648}P + \frac{1}{5444517870735015415413993718908291383296}P - \frac{1}{5444517870735015415413993718908291383296}P + \frac{1}{10889035741470030830827987437816582766592}P - \frac{1}{10889035741470030830827987437816582766592}P + \frac{1}{21778071482940061661655974875633165533184}P - \frac{1}{21778071482940061661655974875633165533184}P + \frac{1}{43556142965880123323311949751266331066368}P - \frac{1}{43556142965880123323311949751266331066368}P + \frac{1}{87112285931760246646623899502532662132736}P - \frac{1}{87112285931760246646623899502532662132736}P + \frac{1}{174224571863520493293247799005065324265472}P - \frac{1}{174224571863520493293247799005065324265472}P + \frac{1}{348449143727040986586495598010130648530944}P - \frac{1}{348449143727040986586495598010130648530944}P + \frac{1}{696898287454081973172991196020261297061888}P - \frac{1}{696898287454081973172991196020261297061888}P + \frac{1}{1393796574908163946345982392040522594123776}P - \frac{1}{1393796574908163946345982392040522594123776}P + \frac{1}{2787593149816327892691964784081045188247552}P - \frac{1}{2787593149816327892691964784081045188247552}P + \frac{1}{5575186299632655785383929568162090376495104}P - \frac{1}{5575186299632655785383929568162090376495104}P + \frac{1}{11150372599265311570767859136324180752990208}P - \frac{1}{11150372599265311570767859136324180752990208}P + \frac{1}{22300745198530623141535718272648361505980416}P - \frac{1}{22300745198530623141535718272648361505980416}P + \frac{1}{44601490397061246283071436545296723011960832}P - \frac{1}{44601490397061246283071436545296723011960832}P + \frac{1}{89202980794122492566142873090593446023921664}P - \frac{1}{89202980794122492566142873090593446023921664}P + \frac{1}{178405961588244985132285746181186892047843328}P - \frac{1}{178405961588244985132285746181186892047843328}P + \frac{1}{356811923176489970264571492362373784095686656}P - \frac{1}{356811923176489970264571492362373784095686656}P + \frac{1}{713623846352979940529142984724747568191373312}P - \frac{1}{713623846352979940529142984724747568191373312}P + \frac{1}{1427247692705959881058285969449495136382746624}P - \frac{1}{1427247692705959881058285969449495136382746624}P + \frac{1}{2854495385411919762116571938898990272765493248}P - \frac{1}{2854495385411919762116571938898990272765493248}P + \frac{1}{5708990770823839524233143877797980545530986496}P - \frac{1}{5708990770823839524233143877797980545530986496}P + \frac{1}{11417981541647679048466287755595961091061972992}P - \frac{1}{11417981541647679048466287755595961091061972992}P + \frac{1}{22835963083295358096932575511191922182123945984}P - \frac{1}{22835963083295358096932575511191922182123945984}P + \frac{1}{45671926166590716193865151022383844364247891968}P - \frac{1}{45671926166590716193865151022383844364247891968}P + \frac{1}{91343852333181432387730302044767688728495783936}P - \frac{1}{91343852333181432387730302044767688728495783936}P + \frac{1}{182687704666362864775460604089535377456991567872}P - \frac{1}{182687704666362864775460604089535377456991567872}P + \frac{1}{365375409332725729550921208179070754913983135744}P - \frac{1}{365375409332725729550921208179070754913983135744}P + \frac{1}{730750818665451459101842416358141509827966271488}P - \frac{1}{730750818665451459101842416358141509827966271488}P + \frac{1}{1461501637330902918203684832716283019655932542976}P - \frac{1}{1461501637330902918203684832716283019655932542976}P + \frac{1}{2923003274661805836407369665432566039311865085952}P - \frac{1}{2923003274661805836407369665432566039311865085952}P + \frac{1}{5846006549323611672814739330865132078623730171904}P - \frac{1}{5846006549323611672814739330865132078623730171904}P + \frac{1}{11692013098647223345629478661730264157247460343808}P - \frac{1}{11692013098647223345629478661730264157247460343808}P + \frac{1}{23384026197294446691258957323460528314494920687616}P - \frac{1}{23384026197294446691258957323460528314494920687616}P + \frac{1}{46768052394588893382517914646921056628989841375232}P - \frac{1}{46768052394588893382517914646921056628989841375232}P + \frac{1}{93536104789177786765035829293842113257979682750464}P - \frac{1}{93536104789177786765035829293842113257979682750464}P + \frac{1}{187072209578355573530071658587684226515959365500928}P - \frac{1}{187072209578355573530071658587684226515959365500928}P + \frac{1}{374144419156711147060143317175368453031918731001856}P - \frac{1}{374144419156711147060143317175368453031918731001856}P + \frac{1}{748288838313422294120286634350736906063837462003712}P - \frac{1}{748288838313422294120286634350736906063837462003712}P + \frac{1}{1496577676626844588240573268701473812127674924007424}P - \frac{1}{1496577676626844588240573268701473812127674924007424}P + \frac{1}{2993155353253689176481146537402947624255349848014848}P - \frac{1}{2993155353253689176481146537402947624255349848014848}P + \frac{1}{5986310706507378352962293074805895248510699696029696}P - \frac{1}{5986310706507378352962293074805895248510699696029696}P + \frac{1}{11972621413014756705924586149611790497021399392059392}P - \frac{1}{11972621413014756705924586149611790497021399392059392}P + \frac{1}{23945242826029513411849172299223580994042798784118784}P - \frac{1}{23945242826029513411849172299223580994042798784118784}P + \frac{1}{47890485652059026823698344598447161988085597568237568}P - \frac{1}{47890485652059026823698344598447161988085597568237568}P + \frac{1}{95780971304118053647396689196894323976171195136475136}P - \$

QUESTION LXXV.

Three persons, C, B, and A, agree to purchase a lease of lands, for the longest of their three lives; the profits of which are to be, equally, divided between the two persons, C and B, during their joint lives; upon the death of either, to be equally divided between C and A, or B and A, the two remaining persons; and lastly, to be enjoyed wholly by the survivor: what ought each person to contribute toward the purchase?

SOLUTION.

By proceeding in the methods above pursued, it will appear, that

CASE I.

If, C and B, the two persons who come first into possession, be younger than A, the person in expectation. (Putting t , m , and n , for the complements of C, B, and A) first,

$$\frac{1}{2} \times \frac{t}{4t} + \frac{1}{4t} \times \frac{t}{2} - \frac{m}{6r} \times m - \frac{n}{6r} \times \frac{2t-n \times n}{2n},$$

will be the value of C's interest.

Secondly,

$$\frac{1}{2} \times \frac{t}{4t} + \frac{1}{4t} \times \frac{t}{2} - \frac{m}{6r} \times m - \frac{n}{6r} \times \frac{2m-n \times n}{2n},$$

will be the value of B's interest.

Thirdly, $\frac{n}{6r} \times \frac{t+m \times n}{4mt}$, will be the value of A's interest.

CASE II.

If, B and C, the two possessors, be one elder, and the other younger, than the expectant, A.

Vol. III.

Q

(Putting

$$\begin{aligned}
 A &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4} \times \frac{m}{4m} \\
 B &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4} \times \frac{m}{4m} \\
 C &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4} \times \frac{m}{4m}
 \end{aligned}$$

COROLLARY II.

If, A and B, the two elder be of equal ages; then $\frac{m}{r} = \frac{m}{r}$, and $m = r$; whence the interest of

$$\begin{aligned}
 A &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4} \\
 B &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4} \\
 C &= \frac{1}{2} \frac{m}{r} + \frac{1}{2} \frac{m}{r} - \frac{1}{2} \frac{m}{r} \times \frac{1}{4}
 \end{aligned}$$

QUEST.

QUESTION LXXV.

Three persons, *C*, *B*, and *A*, agree to purchase a lease of lands, for the longest of their three lives; the profits of which are to be, equally, divided between the two persons, *C* and *B*, during their joint lives; upon the death of either, to be equally divided between *C* and *A*, or *B* and *A*, the two remaining persons; and lastly, to be enjoyed wholly by the survivor: what ought each person to contribute toward the purchase?

SOLUTION.

By proceeding in the methods above pursued, it will appear, that

CASE I.

If, *C* and *B*, the two persons who come first into possession, be younger than *A*, the person in expectation. (Putting *t*, *m*, and *n*, for the complements of *C*, *B*, and *A*) first,

$$\frac{1}{2} + \frac{1}{4t} \times \frac{m}{6r} \times m - \frac{n}{6r} \times \frac{2t-m \times n}{2m},$$

will be the value of *C*'s interest.

Secondly,

$$\frac{1}{2} + \frac{1}{4t} \times \frac{m}{6r} \times m - \frac{n}{6r} \times \frac{2m-n \times n}{2m},$$

will be the value of *B*'s interest.

Thirdly, $\frac{n}{6r} \times \frac{t+m \times n}{4mt}$, will be the value of *A*'s interest.

CASE II.

If, *B* and *C*, the two possessors, be one elder, and the other younger, than the expectant, *A*.

VOL. III.

Q

(Putting

CASE III.

If the possessor C, be elder than the two expectants, A and B. Put x , y , and t , for the complements of C, B, and A; then the interest of the three persons will be,

$$C = \frac{m}{or} \times \frac{m}{4t} + \frac{n}{or} \times \frac{2t - 2m + n \times n}{8mt}$$

$$B = \frac{m}{or} \times \frac{m}{4t} + \frac{n}{or} \times \frac{m}{4t} + \frac{n}{or} \times \frac{2m - 2t + n \times n}{8mt}$$

$$A = \frac{m}{or} \times \frac{m}{4t} + \frac{n}{or} \times \frac{m}{4t} + \frac{n}{or} \times \frac{2m - 2t + n \times n}{8mt}$$

QUESTIONS

An estate of 1000 per annum is leased on two lives (whose complements are x and m) the income of which is to be received by A, so long as both lives continue in being; but is (upon the first demise) to belong to B during the continuance of the remaining life; what is the present value of the interest of B in that lease?

SOLU.

SOLUTION.

From the value of the longest of the two lives $\mathfrak{M} - \frac{1}{2}$

$$\mathfrak{M} - \frac{n}{6r} \times \frac{n}{2m},$$

Take the value of the joint lives $\mathfrak{M} -$

$$\mathfrak{M} - \frac{n}{6r} \times \frac{n}{2m};$$

The remainder will be $\mathfrak{M} - \mathfrak{M}$

$$+ \mathfrak{M} - \frac{n}{6r} \times \frac{n}{m}, \text{ the value required.}$$

And, if the two lives are of the ages of 10 and 39;
Then 7,197 will express that value.

QUESTION LXXVIII.

As estate of 1*l.* per annum is leased for three lives (whose complements are *s*, *m*, and *n*) the income of which is to belong to *A*, so long as all the three lives are in being; but is (upon the first demise) to become the property of *B*, during the continuance of the longest of the remaining two lives; what is the present value of *B*'s interest in that lease?

SOLUTION.

$\mathfrak{S} - \mathfrak{M} + \mathfrak{M} - \frac{m}{6r} \times \frac{m}{2t} + \mathfrak{M} - \frac{n}{6r} \times \frac{t+m}{2mt} n$, will be the value required; and, if the three lives are of the ages, 10, 39, and 48; then 10,775 will express that value.

QUESTION LXXIX.

An estate of 1*l.* per annum is leased for three lives (whose complements are *t*, *m*, and *n*) the income of which

344 MATHEMATICAL

life whose value is \mathcal{R} , will be $P - \mathcal{R}$: but in this case, upon the death of the first tenant, the lord is entitled to the perpetuity of the interest of the fine H ; therefore the present value of the fine, to be received at the admission of the next tenant, will be $\frac{P - H}{1 - d} \times dH$; that is $\frac{1 - d\mathcal{R}}{1 - d} \times H$.

Now, if \mathcal{R} be conceived to be the present value of an annuity, for m years certain; then (by the process used in the scholium to question 89. vol. 2.) it will appear, that

$$1 - d\mathcal{R} = 1 - \frac{dH}{1 - d}$$

But (by the last question) the perpetual recurrency of the fine H , to be received at the expiration of every m years, will be $\frac{H}{1 - d^m}$; that is H divided by $\frac{1 - d^m}{1 - d}$,

which quotient will be $\frac{1 - d\mathcal{R}}{1 - d} \times H$, the worth of the lord's interest in the copyhold for ever, immediately after he has received the admission fine.

And since this present worth, $\frac{1 - d\mathcal{R}}{1 - d} \times H$, will purchase a perpetuity, equal to its interest, for one year (which may be found by multiplying it by d , the interest of 1*l.* for 1 year) therefore the rentroll of the lord's estate should be increased by $\frac{1 - d\mathcal{R}}{1 - d} \times H$, on account of this copyhold.

EXAMPLE.

Suppose that for a certain copyhold 100*l.* fine is paid on every admission, and that the tenants admitted thereto are (one with another) of the age of 25 years; what is the perpetual recurrency of those fines worth in present money, and how much should the rent roll of the lord's estate be increased, on account of those fines; allowing compound interest at 4 per cent?

The

The value of a life of 25 years, secured by land, is 15,504; which being multiplied by 0,04, produces 0,62016.

If $(1 - 0,62016 =) 0,37984$, be multiplied by 100, the product will be 37,984.

Now 37,984, divided by 0,62016, quotes 61,25, for the present value, required; and 37,984, divided by 15,504, quotes 2,45, for the increase of the rent-roll.

But there are two cases, which, though not so generally useful, yet are to be considered, viz.

1st. What the present worth of the lord's interest is, immediately before the receipt of a fine, on admission; and

this is, evidently $(\frac{1 - d^{10}}{d^{10}} H + H =) \frac{H}{d^{10}}$

Secondly. What that present worth will be, at any time after such an admission, when the present tenant's life is worth less than 10.

Put the value of the present tenant's life = 10; then will the fine, to be received at the next admission, be worth $1 - d^{10} \times H$; which is to be added, to the value of perpetual the recurrency of all the usual fines, except the fiftha whence

$\frac{1 - d^{10}}{d^{10}} + 1 - d^{10} \times H$, will be the answer.

QUESTION LXXXIII.

Supposing that the custom of a manor is such, that a tenant at his admission is to pay, to the lord, the sum H , as a fine; what will the present value of the tenant's interest be, in a copyhold estate of 1*l.* per annum, clear of all quitrents, &c?

SOLUTION.

From P the present value of the perpetuity of 1*l.* per ann. take the lord's interest in the copyhold, found by

Q 5 the

after the lease granted and fine paid, the lessor's interest will be $\frac{1-dM}{dM} H$; and the lessee's $P = \frac{1-dM}{dM} H$;

Upon the death of the lessee, before the renewal, and payment of the fine,

The lessor's interest will be $\frac{H}{dM}$; and the lessee's heirs,

$$P \times \frac{M-H}{M}$$

And at any other time,

The Lessor's, $\frac{1-dM^2}{dM} + 1-dM \times H$; and the lessee's

$$P = \frac{1-dM^2}{dM} + 1-dM \times H.$$

But, notwithstanding the same expression answers the above 3 cases, yet the age, whose life is represented by M or M^2 , will in different cases, be very different; for in copyholds, it commonly happens that the tenant to be admitted succeeds his father, grandfather, uncle, brother, &c. as in freehold estates; and consequently a person of any age whatever may be admitted: but none can be presented to a living, till they are above twenty, and consequently (in the case of a perpetual advowson) the mean age will be elder than in the case of the copyhold: and in a leasehold, for lives, it may be presumed, that the best age possible will always be put in; at least, the lessor's fine should be proportioned, in the same manner, as if it were so; and therefore M may, in that case, be generally esteemed to be the value of a life of 10 years: but in the other two cases, the age will be best fixed from observations.

N. B. An ingenious friend of the author's thinks that the ages of the incumbents, when they are inducted, may be partly fixed by the value of the livings; thus, for livings of a 100, 200, 300, and 400*l.* per ann. at 45, 34, 42, and 50 years respectively.

QUES-

QUESTION LXXXV.

There is an estate of 1L . per ann. which is leased on two lives; with condition, that when both lives are extinct, the same may be renewed for two other such lives, on paying the fine K ; the interests of the lessor and lessee, in that estate, are required.

SOLUTION.

If L denote the value of an annuity on the longest of the two lives, then by arguing, as in quest. 82; the interest of the lessor immediately after signing the lease, and receiving the fine, will be $\frac{1-dL}{dL} K$;

the rent-roll of the lessor's estate, ought to be increased by $\frac{1-dL}{L} K$, on account of the perpetual recurrence of these fines; the interest of the lessor at the extinction of the second life, and before receiving the fine on renewal will be $\frac{K}{dL}$; and at any other time, when

the lives or life in possession are worth but 10 year's purchase, then the interest of the lessor will be $\frac{1-dL^{10}}{dL^{10}} + 1-d10 \times K$: the interest of the lessee

being in each case, the difference between the perpetuity and the value so found.

COROLL.

Hence, the values of the respective interests of the lessor and lessee, of any lease for lives, renewable only at its expiration, may be found, by using the value of the lives, after the manner above expressed.

QUEST.

SOLUTION.

It is evident, that the lessor (by granting the above lease) alienates the whole perpetuity, except the value of those fines; now it may be obtained by arguing as in questions 82 and 85, that the present worth of all the

fines denoted by H , will be $(\frac{1 - d^{38}}{d} H =) 5,798$

and the present worth of all the fines denoted by K will be $(\frac{1 - d^{38}}{d} K =) 6,052$.

Now, since the lessee (or his heirs) are at liberty, either to renew at the first, or second death; and, in case the one is constantly performed, the lessor will not be, at all, entitled to the other; therefore the mean of the values

of the two fines, viz. $(\frac{5,798 + 6,052}{2} =) 5,925$, may

be considered as the value which he has reserved: and consequently $(25 - 5,925 =) 19,075$ will be the fine, which ought to be paid on granting the first lease.

S C H O L I U M,

If the fines H and K , could have been proportioned with the desired accuracy, their values would have been equal: but the inequality has two causes; the first is, that the age 38 is (in quest. 86.) found without any regard to the interest of money; the second is, that the said age, 38, is not exact, tho' the nearest whole number thereto; the true age (upon those principles) being $(86 - 48,18 =) 37,82$.

QUESTION LXXXIX.

There is a lease for two lives; which may, on the first demise, be filled up with a life, whose complement is t ,
on

on paying a fine certain, H ; and when both lives are extinct, it may be renewed for two lives, whose complements are r , and m , on paying the fine K : Now, one of the lives having failed, and the complement of the survivor being c , it is required to determine, whether it is the lessee's interest to renew the lease (by putting in a life) now; or to stay till that life is also extinct, and then to put in two lives?

SOLUTION.

If the lessee were to renew at the time that each single life fails, he must then pay the fine, H , again, when the joint lives, whose complements are r , and c , shall fail; but if he constantly renews at the expiration of the second life, he then pays the fine, K , again, when the longest of the two lives, whose complements are r and m shall fail; whence (by quest. 82) the interest of the lessor, immediately before those renewals, will (in the first case) be $\frac{H}{d \cdot \mathcal{L}}$; and (in the second) $\frac{K}{d \cdot L}$: but, if we value those two interests at the failure of the first life, the latter (not being due till the death of the life in being, whose value is \mathcal{L}) will be then worth, but

$$1 - d \cdot \mathcal{L} \times \frac{K}{d \cdot L}, \text{ by quest. 89. vol. 2.}$$

Now, if the fines, H and K , are so proportioned, as to make it indifferent to the lessee, whether he renews at the first or second demise; then $\frac{H}{d \cdot \mathcal{L}} = \frac{1 - d \cdot \mathcal{L} \times K}{d \cdot L}$:

$$\text{or } \frac{H}{d \cdot \mathcal{L}} = \frac{1 - d \cdot \mathcal{L} \times K}{d \cdot L}$$

But, when the fines are proportioned as in the last quest. on, $L \cdot \mathcal{L} = K$; whence $\frac{H}{d \cdot \mathcal{L}} = 1 - d \cdot \mathcal{L}$.

From

356 MATHEMATICAL

will be the expectation of the life surviving those two ; which, because the lease, then made, is to be of equal value with the original, will be a life whose complement is n , and expectation $\frac{n}{2}$; therefore,

$$\frac{t}{2} - \frac{m}{2} + \frac{2mm}{6t} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{3n^3}{12mt} = \frac{n}{2}.$$

Or

$$\frac{mm}{6t} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{3n^3}{12mt} = \frac{m}{2} + \frac{n}{2} - \frac{t}{2}$$

$$\left(- \frac{mm}{6t} \right)$$

Now by comparing the two equations together, it will appear, that,

$$\frac{mm}{6t} - \frac{nn}{6m} - \frac{nn}{6t} + \frac{3n^3}{12mt} = \frac{nn}{6t}$$

That is, $2m^3 - 2mm - 4mn + 3n^3 = 0$;
This equation (if we write $(86 - 10 =) 76$ for t) becomes
 $2m^3 - 152mm - 4mn + 3n^3 = 0$;

Where the whole numbers 58, and 38, are (nearly) equal to the values of m , and n ;

Therefore, the two ages required will be $(86 - 38 =) 48$, and $(86 - 38 =) 48$; and (when a renewal is to be made, after two of the three lives have failed) no 2 lives should be put in, the expectation on the longest of which exceeds $\frac{76}{2} + \frac{58 \times 58}{6 \times 76} = 45.38$.

QUESTION XCI.

It is proposed to grant a lease of an estate of 1*l*. per annum, for three lives, of the respective ages of 10, 28, and 48; with a condition, that the lessee (or his heirs) may on every first failure, put in a new life, aged 10 years, on paying a fine certain: what sum ought that fine to be fixed at, in the original lease?

S O L U.

SOLUTION

The present value of the reversion of a younger life after two elder lives is investigated in case 3. quest. 29, and by applying the result to the present case it will appear, that 2,676 £ . will be the fine required.

QUESTION XCII.

It is proposed to grant a lease of an estate of 1 £ . per annum, for three lives, of the respective ages of 10, 28, and 48, with a condition, that the lessee (or his heirs) may, on every failure of two lives, put in two new lives, aged 10 and 28 years, on paying a fine certain; that fine is required?

SOLUTION.

The present value of the reversion of the longest of 2 younger lives after an elder life, is found in case 3. quest. 28, which, being applied to the present case, will give 7,508 for the fine required.

QUESTION XCIII.

It is proposed to grant a lease of an estate of 1 £ . per annum, for three lives, aged 10, 28, and 48; with conditions that the lessee (or his heirs) may, perpetually, renew the same; either, at the first death, for a fine of 2,676; or, when two of the lives have failed, for 7,508; or, when all three are extinct, for 19,510 (the value of the longest of the three lives): what fine ought to be paid, on granting the original lease?

SOLU

SOLUTION.

Let x represent the value of the three joint lives; y the value of the longest of any two, out of the three; L the longest of the three lives; H , the fine 2,676; K , the fine 7,508; and G , the fine 19,510.

Then the present worth of all the fines, denoted by H , will be $\frac{1-d}{d}$; of all the fines, denoted by K ,

$\frac{1-d}{d}$; and of all the fines, denoted by G , $\frac{1-d}{d}$.

whence, by calculation, first, 4,760 will be the value of all the fines, denoted by H .

Secondly, 4,732 will be the value of all the fines, denoted by K .

And thirdly 5,490 will be the value of all the fines, denoted by G .

Then (by arguing as in question 88,

$\frac{4,700 + 4,732 + 5,490}{3} = 4,974$ will be the value

of the lessor's interest in all the fines; and $(25 - 4,974 =) 20,026$, the fine to be paid, on granting the original lease.

QUESTION XCIV.

There is an estate of 1*l.* per annum, which is leased, on three lives; with conditions, that (when a single life drops) another, whose complement is t , may be put in, on paying the fine H ; if two lives are extinct, then two other

other lives, whose complements are t and m , may be added, on paying the fine K ; and, if all the three lives are expired, then a new lease for three lives, whose complements are t , m , and n , shall be granted, on paying the fine G : Let it now be supposed, that one of the three lives is extinct, and that the complements of the two remaining lives are c and b ; it is required to determine, whether it will be the lessee's interest to put in a life at this first demise, or to stay till the second, and then to put in two?

SOLUTION.

If the lessee were to renew, at the time that each single life fails, he then must pay the fine, H , again, when one of the three lives whose complements are t , c , and b , shall fail; but, if he constantly renews at the second death, he is then to pay the fine, K , a second time, when two lives out of the three, whose complements are t , m , and c , shall fail; and then c the complement of the survivor of the lives whose complements are c and b , will (by the process in quest. 86, probably) be worth

$$\left(2 \times \frac{c}{2} - \frac{b}{2} + \frac{2bt}{6c}\right) c - b + \frac{2bt}{3c}.$$

Now, in the first case, the interest of the lessor, immediately before those renewals, will be $\frac{H}{d \cdot \overline{t|c|b}}$; and, in

the second case, it will be $\frac{K}{d \cdot \overline{t|m|c}}$; the latter of which (because the fine K is not payable, till the failure of one of the lives, whose complements are c and b) will be, at the time of the first demise, worth, only,

$$\frac{K}{d \cdot \overline{t|c|b}} \times \frac{K}{d \cdot \overline{t|m|c}}.$$

Now, if the fines, H and K , are so proportioned, as to make it indifferent to the lessee, whether he renews at the

the first, or second demise; then,

$$\frac{H}{1 - \frac{1}{100} \times K} = \frac{1}{1 - \frac{1}{100} \times K}$$

Whence, if $\frac{H}{1 - \frac{1}{100} \times K}$ exceeds $\frac{H}{1 - \frac{1}{100} \times K}$, then it

will be the lessor's interest, that the lease should not be renewed till the next life-fails; and consequently the lessee's

to renew now; but, if $\frac{H}{1 - \frac{1}{100} \times K}$ exceeds $\frac{1 - \frac{1}{100} \times K}{1 - \frac{1}{100} \times K}$,

then it will be the lessor's interest to have the lease renewed now; and the lessee's, to postpone it.

Note. When the ages and fines are properly proportioned, as in questions 90, 91, 92, and 93; if, on the failure of the first life, the joint lives of the two survivors is worth more, than the joint lives of two persons, aged 28 and 48, it will be the lessee's interest to postpone the renewal, till one of them fails; but if the joint lives of 28 and 48, are worth more, than the joint lives of the survivors, it will be his interest to renew directly; which is evident, from the solutions of those questions.

QUESTION XCV.

In such a lease as that mentioned in the last question; let us suppose, that two, of the three lives, have failed, the complement of the survivor being c ; Then, will it be the interest of the lessee to renew directly, or to stay till the remaining life is also extinct.

SOLUTION.

If the lessee were to renew constantly on the failure of the second life, he is to pay the fine, K , a second time, when two lives, out of the three whose complements are

are t , m , and c , shall fail; but if he constantly renews when all the three lives are extinct, then he will be to pay the fine, G , a second time, upon the failure of all the three lives, whose complements are t , m , and n .

Then, in the first case, the interest of the lessor, immediately before the renewals will be $\frac{K}{d \times \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$, and

$\frac{G}{dL \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$; the latter of which (because the fine G is not payable till the failure of the life whose complement is c) will, at the time of the second demise, be worth, only, $\frac{1-d \times G}{dL \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$.

Now, if the fines, K and G , are so proportioned, as to make it indifferent to the lessee, whether he renews at the second or third death; then,

$$\frac{K}{d \times \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}} = \frac{1-d \times G}{dL \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}.$$

Whence, if $\frac{1-d \times G}{L \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$ exceeds $\frac{K}{\frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$, then it will be the lessor's interest, that the lease should not be renewed till the third life fails; and, consequently, the lessee's to renew directly: But, if $\frac{K}{\frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$ exceeds

$\frac{1-d \times G}{L \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d}}$, then it will be the lessor's interest to have the lease renewed now, and the lessee's to postpone it.

Note. When the ages and fines are properly proportioned, as in questions 90, 91, 92, and 93; then $L \cdot \frac{1-d}{1-d} \times \frac{1-d}{1-d} \times \frac{1-d}{1-d} = G$; and, if the remaining life (whose complement is c) be younger than 48; it will be the interest of the lessee not to renew, till the failure thereof; but if the survivor is elder than 48. then to renew directly.

QUESTION XCVI.

What sum ought to be paid for insuring, for one year, a given sum, s , on a life, whose complement is n ?

SOLUTION,

Since the probability of the life's failing in that year is $\frac{1}{n}$, and (if that happens) the insurer is to pay to the insured, the sum, s ; therefore, the premium, for that insurance, should be $\frac{s}{n}$.

EXAMPLE.

If the sum of 100 £. is insured on a life, aged sixty six years; then $n = (86 - 66) = 20$; and $(\frac{100}{20} =) 5$ £. will be the required premium.

Note. The premium of insurance is usually paid down, at the time of insuring; but the sum insured does (in case of the death of the person whose life is insured) generally remain unpaid, till the end of the year, and sometimes longer; now (if the computation be made, as above) it is evident, that the premium is but just, adequate to the hazard: and consequently, if the insurer is to expect any profit, on the transaction; the interest of the premium (which by the above custom he enjoys) for one year, will be a very moderate one; being (in the above example) but 5 s. if he could make 5 £. per cent. of his money.

QUESTION XCVII.

What constant yearly sum, of which the first payment is to be made immediately, ought to be paid, for insuring the sum, s , on a life whose complement is n ?

SOLU.

SOLUTION.

By the last question, the premium for insuring the first year, will be $\frac{s}{n}$; and, when that is elapsed, then, for the second year, $\frac{s}{n-1}$; for the third, $\frac{s}{n-2}$, &c.

But, forasmuch as the premium, for the second, third, &c. year, will not be payable, unless the insured survives the first, second, &c. year; the probabilities of which are $\frac{n-1}{n}$, $\frac{n-2}{n}$, &c. therefore, the premiums for the second, third, &c. year, being estimated at the beginning of the first year, will be $\frac{n-1}{n} \times \frac{s}{n-1}$, $\frac{n-2}{n} \times \frac{s}{n-2}$, &c. or $\frac{s}{n}$, $\frac{s}{n}$, &c.

Again, because the said premiums for the 2d. 3d. &c. year, do not become payable, till the expiration of the 1st. 2d. &c. year; the present worth of them, computed at the beginning of the first year, will be $\frac{s}{nr}$, $\frac{s}{nr^2}$, &c. and, since one payment is to be made immediately, therefore

$$\left(\frac{s}{n} + \frac{s}{nr} + \frac{s}{nr^2} (n) = \right)$$

$$\frac{s}{n} \times \frac{1}{1} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n),$$

will be the present worth of all the yearly premiums, that can be received by the insurer; for, by the hypothesis of equal decrements, the life will be necessarily extinct, at the end of the n th year; and consequently, $\frac{s}{nr^{n-1}}$

will be the present worth of the last premium, that he can possibly receive.

364 MATHEMATICAL

Now $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n) = (r \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n)) =$
 $\frac{r}{r-1} \times 1 - \frac{1}{r^n}$ (by quest. 15. vol. 2.) therefore (putting
 $\frac{1}{r-1} = P$, and $\frac{1}{r^n} = p$) $\frac{s}{n} + \frac{s}{nr} + \frac{s}{nr^2} (n)$ the pre-
 sent worth of all the unequal premiums, which can be re-
 ceived by the insurer, will be $\frac{srP}{n} \times \frac{1}{1-p}$.

But the present worth of all the constant annual sums,
 which the insurer is to receive, for insuring the life in
 question, will (if N represents the value of a life, not
 secured by land, whose complement is n ; and a , the year-
 ly sum, so to be received) be $(a + Na)$ or $\frac{1}{1+r} \times a$;
 for the sum, a , is to be immediately received, and the
 receipt thereof is to be annually continued, during the
 life, whose value is N ; also, if the life fails, in any part of
 a year, no sum is to be received at the end of that year;
 for then, the sum insured is to be paid to the insured, or
 his assigns.

It follows, therefore, that $\frac{srP}{n} \times \frac{1}{1-p}$, the present worth
 of all the yearly unequal premiums, which the insurer
 would receive, if the life was to be annually insured;
 will be equal to, $\frac{1}{1+r} \times a$, the present worth of all the
 equal premiums, which he is to receive, if he engages to
 insure the life, on the receipt of a constant yearly sum:

That is $1 + N \times a = \frac{srP}{n} \times \frac{1}{1-p}$;

Therefore $a = \frac{srP}{n} \times \frac{1-p}{1+N}$.

Example. What constant annual premium should be
 paid, for insuring 100*l.* on a life, aged 66, allowing 4
 per cent. compound interest?

Here $s=100$; $r=1.04$; $P=25$; $n=(86-66)=20$;
 $p=0.4564$; $1-p=0.5436$; $N=7.333$; and $1+N=$
 8.333 : There-

Therefore $\left(\frac{100 \times 1.04 \times 25 \times 0.5436}{20 \times 8.333} \right) = 8.481$ will be the annual premium required.

Scholium. The above question may be, otherwise, solved, in the following manner: the insured purchases the sum s , to be paid to him, or his assigns, on the decease of a life, whose value is N ; by the payment of the sum, a , in hand, and by the annual payment of the like sum, a , during the continuance of the said life.

Now (by quest 89. vol. 2.) the present worth of 1*l.* payable on the failure of the proposed life, is

$$\frac{1-p \times r}{n \times r-1}, \text{ or } \frac{1-r-1}{n} \times N; \text{ and, consequently, the pre-}$$

sent value of s will be equal to $\frac{1-p \times r P_s}{n}$ (because $\frac{1}{r-1}$

$$= P), \text{ or } \frac{1-r-1}{n} \times V \times s;$$

$$\text{But } 1 + N \times a = \left(\frac{1-p \times r P_s}{n} \text{ or } \frac{1-r-1}{n} \times N \times s; \right)$$

$$\text{Therefore } a = \left(\frac{1-p \times r P_s}{1+N \times n} \text{ or } \frac{1-r-1 \times N \times s}{1+N} \right)$$

In the above example; $r-1 = 0.04$; $7.333 \times 0.04 = 0.29332$; $1 - 0.29332 = 0.70668$; and $0.70668 \times 100 = 70.668$:

Therefore $\left(\frac{70.668}{8.333} \right) = 8.481$ will be the annual payment, as before.

QUESTION XCVIII.

A. who has a place, or other annual income for life, will borrow a sum of money of *B*, and (besides a bond, bearing interest) would insure his life, as a collateral security; *B* undertakes to be the insurer, and to remit the payment of the principal, to the heirs of *A*, if he constantly pays the said interest and insurance, during his

366 MATHEMATICAL

life to him, B , or his heirs; what sum ought A to pay, annually for interest and insurance, for 1*£*. so lent?

SOLUTION.

Let N be the value of A 's life; then (by the scholium to quest. 97.) the annual insurance of 1*£*. will be

$$\frac{1-r-1 \times N}{1+N}; \text{ to which add, } r-1, \text{ the interest of 1*£*.}$$

And the sum $\left(\frac{1-r-1 \times N}{1+N} + r-1 \right) = \frac{r}{1+N}$ will be the annual payment required.

Example. If A , the borrower, be aged 66; and interest be computed at 4 per cent; then $r = 1.04$, and $1+N = 8.333$:

Whence $\left(\frac{1.04}{8.333} = \right) 0.125$ will be the sum required.

SCHOLIUM.

It is necessary, here, to observe, that the borrower ought not to receive the full sum of 1*£*. (here supposed to be lent) because he is to pay the first year's insurance down; and, consequently should receive but

$$\begin{aligned} \left(1 - \frac{1-r-1 \times N}{1+N} = \frac{1+N-1+r-1N}{1+N} \right. \\ \left. = \frac{N+r-1 \times N}{1+N} = \right) \frac{rN}{1+N}; \text{ in the above example} \\ \left(\frac{1.04 \times 7.333}{8.333} = \right) 0.915. \end{aligned}$$

This will appear to be true, by considering that the sum,

$\frac{rN}{1+N}$ will purchase an annuity of, $\frac{r}{1+N}$, for a life

whose value is N ; for if $N a = \frac{rN}{1+N}$; then $a = \frac{r}{1+N}$.

A TABLE of the present values of an annuity (secured by land) of 1*l.* per annum, on a single life; supposing the decrements of life to be equal.

Ages	3 per cent	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent
8	19,934	18,330	16,949	15,738	14,674	12,899
9	20,064	18,443	17,038	15,813	14,735	12,947
10	20,064	18,443	17,038	15,813	14,735	12,947
11	19,934	18,336	16,949	15,738	14,674	12,899
12	19,804	18,227	16,858	15,661	14,611	12,852
13	19,671	18,117	16,765	15,583	14,545	12,804
14	19,535	18,005	16,671	15,503	14,477	12,753
15	19,398	17,891	16,575	15,422	14,407	12,702
16	19,258	17,774	16,478	15,340	14,335	12,649
17	19,115	17,655	16,378	15,255	14,263	12,595
18	18,971	17,534	16,276	15,170	14,189	12,539
19	18,824	17,412	16,172	15,080	14,114	12,483
20	18,675	17,287	16,066	14,990	14,036	12,424
21	18,524	17,159	15,958	14,898	13,957	12,364
22	18,369	17,029	15,848	14,804	13,876	12,304
23	18,213	16,896	15,736	14,708	13,793	12,240
24	18,053	16,762	15,621	14,609	13,708	12,176
25	17,892	16,624	15,504	14,510	13,622	12,111
26	17,728	16,484	15,386	14,407	13,533	12,042
27	17,560	16,343	15,265	14,302	13,442	11,974
28	17,390	16,198	15,139	14,195	13,348	11,902
29	17,217	16,050	15,012	14,084	13,252	11,829
30	17,041	15,900	14,882	13,973	13,155	11,753
31	16,863	15,747	14,750	13,857	13,054	11,675

A TABLE of the present values of an annuity (secured by land) of 1 £. per annum, on a single life; supposing the decrements of life to be equal.

Ages	3 per cent	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent
32	16,082	15,590	14,615	13,739	12,952	11,997
33	16,497	15,430	14,476	13,619	12,847	11,515
34	16,300	15,268	14,335	13,496	12,739	11,431
35	16,118	15,103	14,191	13,370	12,629	11,344
36	15,923	14,934	14,044	13,242	12,516	11,256
37	15,725	14,762	13,894	13,110	12,399	11,163
38	15,523	14,586	13,740	12,974	12,279	11,070
39	15,319	14,406	13,582	12,837	12,157	10,973
40	15,111	14,225	13,423	12,695	12,031	10,873
41	14,898	14,039	13,258	12,550	11,902	10,771
42	14,683	13,849	13,091	12,401	11,771	10,665
43	14,464	13,656	12,920	12,248	11,635	10,556
44	14,240	13,459	12,744	12,093	11,495	10,444
45	14,014	13,256	12,566	11,933	11,353	10,329
46	13,782	13,051	12,382	11,770	11,207	10,209
47	13,546	12,842	12,195	11,602	11,055	10,087
48	13,308	12,628	12,002	11,430	10,901	9,960
49	13,064	12,409	11,806	11,253	10,741	9,829
50	12,814	12,186	11,607	11,072	10,578	9,695
51	12,562	11,959	11,402	10,887	10,410	9,556
52	12,305	11,727	11,192	10,697	10,237	9,412
53	12,044	11,489	10,977	10,501	10,059	9,265
54	11,776	11,248	10,757	10,301	9,877	9,111
55	11,506	11,000	10,532	10,096	9,688	8,954

A TABLE of the present values of an annuity (secured by land) of 1*l*. per annum, on a single life; supposing the decrements of life to be equal.

Ages	3 per cent	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent
56	11,229	10,750	10,302	9,885	9,495	8,790
57	10,947	10,493	10,066	9,669	9,297	8,621
58	10,660	10,228	9,825	9,447	9,092	8,447
59	10,368	9,960	9,577	9,219	8,882	8,268
60	10,071	9,686	9,324	8,985	8,665	8,081
61	9,767	9,406	9,065	8,746	8,443	7,889
62	9,460	9,121	8,800	8,499	8,214	7,690
63	9,144	8,828	8,528	8,246	7,977	7,488
64	8,824	8,529	8,250	7,985	7,734	7,271
65	8,499	8,225	7,965	7,718	7,484	7,050
66	8,166	7,913	7,673	7,444	7,227	6,822
67	7,827	7,595	7,372	7,162	6,961	6,586
68	7,481	7,268	7,066	6,871	6,686	6,341
69	7,130	6,937	6,752	6,574	6,405	6,087
70	6,771	6,597	6,429	6,269	6,114	5,824
71	6,406	6,249	6,099	5,954	5,814	5,551
72	6,034	5,895	5,760	5,630	5,506	5,269
73	5,655	5,532	5,413	5,298	5,187	4,976
74	5,269	5,162	5,057	4,956	4,858	4,673
75	4,874	4,782	4,691	4,605	4,521	4,358
76	4,473	4,394	4,318	4,243	4,170	4,032
77	4,064	3,998	3,933	3,871	3,810	3,692
78	3,646	3,593	3,540	3,488	3,437	3,341
79	3,221	3,178	3,136	3,094	3,054	2,977

A TABLE of the differences, between the values of annuities for single lives (secured by land) and the quotients arising, from the divisions of the respective complements of those lives, by 6 times the rate of interest; which differences occur, in most of the solutions contained in this volume.

Ages	3 per cent	3 $\frac{1}{2}$ per cent	4 per cent	4 $\frac{1}{2}$ per cent	5 per cent	6 per cent
8	7,759	6,259	4,930	3,776	2,770	1,107
9	7,766	6,205	4,858	3,692	2,672	0,997
10	7,766	6,205	4,858	3,692	2,672	0,997
11	7,799	6,259	4,930	3,776	2,770	1,107
12	7,830	6,311	4,999	3,859	2,865	1,217
13	7,859	6,362	5,066	3,940	2,958	1,326
14	7,885	6,411	5,132	4,020	3,059	1,433
15	7,909	6,458	5,196	4,098	3,137	1,539
16	7,931	6,502	5,260	4,176	3,224	1,643
17	7,950	6,544	5,320	4,250	3,311	1,746
18	7,968	6,584	5,378	4,325	3,395	1,848
19	7,983	6,623	5,435	4,395	3,479	1,948
20	7,996	6,659	5,489	4,464	3,560	2,047
21	8,006	6,692	5,541	4,531	3,639	2,144
22	8,013	6,723	5,591	4,597	3,717	2,241
23	8,019	6,751	5,639	4,660	3,793	2,335
24	8,021	6,778	5,686	4,721	3,867	2,428
25	8,022	6,801	5,728	4,781	3,940	2,520
26	8,019	6,822	5,770	4,838	4,009	2,609
27	8,013	6,842	5,810	4,892	4,077	2,698
28	8,005	6,859	5,844	4,945	4,142	2,783
29	7,994	6,871	5,877	4,994	4,204	2,867
30	7,980	6,883	5,908	5,042	4,266	2,948
31	7,964	6,891	5,936	5,085	4,324	3,027

A TABLE of the differences, between the values of annuities for lives (secured by land) and the quotients arising, from the divisions of the respective complements of those lives, by 6 times the rate of interest; which differences occur, in most of the solutions contained in this volume.

Ages	3 per cent	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent
32	7,944	6,895	5,961	5,126	4,381	3,107
33	7,921	6,896	5,982	5,166	4,434	3,182
34	7,895	6,895	6,001	5,202	4,485	3,255
35	7,866	6,891	6,018	5,236	4,534	3,325
36	7,832	6,883	6,031	5,267	4,580	3,394
37	7,796	6,872	6,041	5,295	4,622	3,459
38	7,756	6,857	6,048	5,319	4,660	3,523
39	7,714	6,838	6,050	5,341	4,697	3,583
40	7,668	6,818	6,051	5,358	4,729	3,640
41	7,617	6,793	6,046	5,373	4,759	3,695
42	7,564	6,764	6,039	5,383	4,787	3,747
43	7,506	6,732	6,029	5,390	4,810	3,795
44	7,444	6,696	6,013	5,394	4,829	3,840
45	7,380	6,654	5,995	5,394	4,845	3,883
46	7,310	6,610	5,972	5,390	4,858	3,920
47	7,235	6,562	5,945	5,382	4,865	3,955
48	7,159	6,509	5,912	5,369	4,869	3,985
49	7,077	6,451	5,876	5,352	4,868	4,011
50	6,989	6,389	5,838	5,330	4,864	4,035
51	6,898	6,323	5,793	5,305	4,854	4,053
52	6,804	6,252	5,743	5,274	4,840	4,066
53	6,704	6,175	5,689	5,238	4,821	4,076
54	6,598	6,095	5,629	5,196	4,798	4,079
55	6,490	6,008	5,565	5,152	4,767	4,080